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Measures on the unit circle and unitary truncations of unitary operators

M.J. Cantero, L. Moral, L. Velázquez*

Departamento de Matemática Aplicada, Universidad de Zaragoza, 50009 Zaragoza, Spain

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Abstract

In this paper, we obtain new results about the orthogonality measure of orthogonal polynomials on the unit circle, through the study of unitary truncations of the corresponding unitary multiplication operator, and the use of the five-diagonal representation of this operator.

Unitary truncations on subspaces with finite co-dimension give information about the derived set of the support of the measure under very general assumptions for the related Schur parameters (a_n). Among other cases, we study the derived set of the support of the measure when $\lim_n |a_{n+1}/a_n| = 1$, obtaining a natural generalization of the known result for the López class $\lim_n a_{n+1}/a_n \in \mathbb{T}$, $\lim_n |a_n| \in (0, 1)$.

On the other hand, unitary truncations on subspaces with finite dimension provide sequences of unitary five-diagonal matrices whose spectra asymptotically approach the support of the measure. This answers a conjecture of L. Golinskii concerning the relation between the support of the measure and the strong limit points of the zeros of the para-orthogonal polynomials.

Finally, we use the previous results to discuss the domain of convergence of rational approximants of Carathéodory functions, including the convergence on the unit circle.

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* Corresponding author.

E-mail addresses: mjcante@unizar.es (M.J. Cantero), lmoral@unizar.es (L. Moral), velazque@unizar.es (L. Velázquez).

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1. Introduction

In [7,8], a new operator theoretic approach for the orthogonal polynomials with respect to a measure on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ was established. The five-diagonal representation of unitary operators introduced there has proved to be a powerful tool for the study of such orthogonal polynomials, as it has been shown in [26,27], where numerous new results have been obtained (for a summary of some of the main new results in [26,27], see [25]). Let us summarize the main facts concerning this five-diagonal representation, since it is the starting point of this paper.

In what follows μ denotes a probability measure on $\mathbb{T}$ with an infinite support supp μ. Then,

$$U^{\mu}: L^2_{\mu} \underset{f(z)\to zf(z)}{\longrightarrow} L^2_{\mu},$$

is a unitary operator on the Hilbert space L^2_{μ} of μ-square-integrable functions with the inner product

$$(f, g) := \int f(z)\overline{g(z)}\, d\mu(z), \quad \forall f, g \in L^2_{\mu}.$$

The associated five-diagonal representation is just the matrix representation of U^{μ} with respect to an orthonormal Laurent polynomial basis $(\chi_n)_{n\geqslant 0}$ of L^2_{μ}. This basis is related to the usual orthonormal polynomials $(\varphi_n)_{n\geqslant 0}$ in L^2_{μ}, defined by

$$\varphi_n(z) = \kappa_n(z^n + \cdots + a_n), \quad \kappa_n > 0, \quad (\varphi_n, \varphi_m) = \delta_{n,m}, \quad n, m \geqslant 0,$$

through the relations [29,7]

$$\chi_{2j}(z) = z^{-j}\varphi^*_{2j}(z), \quad \chi_{2j+1} = z^{-j}\varphi_{2j+1}(z), \quad j \geqslant 0, \tag{1}$$

where, for every polynomial p of degree n, $p^*(z) := z^n\overline{p}(z^{-1})$ is called the reversed polynomial of p. The five-diagonal representation has the form [7]

$$C(\boldsymbol{a}) := \begin{pmatrix} -a_1 & -\rho_1 a_2 & \rho_1\rho_2 & & & & & \\ \rho_1 & -\overline{a}_1 a_2 & \overline{a}_1\rho_2 & 0 & & & & \\ 0 & -\rho_2 a_3 & -\overline{a}_2 a_3 & -\rho_3 a_4 & \rho_3\rho_4 & & & \\ & \rho_2\rho_3 & \overline{a}_2\rho_3 & -\overline{a}_3 a_4 & \overline{a}_3\rho_4 & 0 & & \\ & & 0 & -\rho_4 a_5 & -\overline{a}_4 a_5 & -\rho_5 a_6 & \rho_5\rho_6 & \\ & & & \rho_4\rho_5 & \overline{a}_4\rho_5 & -\overline{a}_5 a_6 & \overline{a}_5\rho_6 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2}$$

where $\boldsymbol{a} := (a_n)_{n\geqslant 1}$ satisfies $|a_n| < 1$ and $\rho_n := \kappa_{n-1}/\kappa_n = \sqrt{1-|a_n|^2}$. The transposed matrix $C(\boldsymbol{a})^t$ of $C(\boldsymbol{a})$ is also a representation of U^{μ}, but with respect to the orthonormal Laurent polynomial basis $(\chi_{n*})_{n\geqslant 0}$, where $f_*(z) := \overline{f}(z^{-1})$ for any Laurent polynomial f.

$C(\boldsymbol{a})$ and $C(\boldsymbol{a})^t$ can be identified with unitary operators on the Hilbert space ℓ^2 of square-sumable sequences in $\mathbb{C}$, these operators being unitarily equivalent to U^{μ}. Due to the properties of the multiplication operator, the spectrum of $C(\boldsymbol{a})$ and $C(\boldsymbol{a})^t$ coincides with supp μ, the mass points being the corresponding eigenvalues. Since the eigenvalues are simple, the essential spectrum of $C(\boldsymbol{a})$ and $C(\boldsymbol{a})^t$ (that is, the spectrum except the isolated eigenvalues with finite multiplicity) is the derived set $\{\text{supp}\,\mu\}'$ of supp μ.

$\boldsymbol{a}$ is called the sequence of Schur parameters of μ. The Schur parameters establish a one to one correspondence between sequences in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and probability measures on $\mathbb{T}$ with infinite support. The Schur parameters also appear in the forward recurrence relation

$$\rho_n \varphi_n(z) = z\varphi_{n-1}(z) + a_n \varphi_{n-1}^*(z), \quad n \geqslant 1, \tag{3}$$

that generates the orthonormal polynomials, which is also equivalent to the backward recurrence relation

$$\rho_n z\varphi_{n-1}(z) = \varphi_n(z) - a_n \varphi_n^*(z), \quad n \geqslant 1. \tag{4}$$

Therefore, the matrix $C(\boldsymbol{a})$ provides a connection between this practical way of constructing sequences of orthonormal polynomials on $\mathbb{T}$, and the properties of the related orthogonality measure μ, lost in such construction. In particular, the spectral analysis of $C(\boldsymbol{a})$ permits us to recover features of $\operatorname{supp}\mu$ from properties of the Schur parameters $\boldsymbol{a}$. Only in the case $|a_n| < 1$ the matrices $C(\boldsymbol{a})$ are related to a measure on $\mathbb{T}$ with infinite support. However, they are well defined unitary matrices even if $|a_n| \leqslant 1$. As we will see, this is important when using perturbative arguments for the analysis of the measure.

In this paper, we use unitary truncations of $C(\boldsymbol{a})$ as a source of information for the spectrum and the essential spectrum of $C(\boldsymbol{a})$, that is, for $\operatorname{supp}\mu$ and $\{\operatorname{supp}\mu\}'$. Let us denote by $\{e_n\}_{n\geqslant 1}$ the canonical basis of ℓ^2, and let $\ell_n^2 := \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. As we will see in the following section, for any infinite bounded normal band matrix, the normal truncations on ℓ_n^2 or $\ell_n^{2\perp}$ have a spectrum closely related to the spectrum of the full matrix. This justifies the study of this kind of normal truncations for $C(\boldsymbol{a})$, which is the aim of Section 2. We find that all these truncations are indeed unitary and can be parameterized by the points in $\mathbb{T}$, leading to the para-orthogonal polynomials [20] and to the family of Aleksandrov measures [16] related to the associated polynomials [23].

In Section 3 we use the unitary truncations of $C(\boldsymbol{a})$ on $\ell_n^{2\perp}$ to obtain general relations between the asymptotic behaviour of the Schur parameters and the location of $\{\operatorname{supp}\mu\}'$. A well known result is that, under the conditions $\lim_n a_{n+1}/a_n \in \mathbb{T}$, $\lim_n |a_n| \in (0,1)$, which define the so-called López class, $\{\operatorname{supp}\mu\}'$ is a closed arc centred at $-\lim_n a_{n+1}/a_n$ with angular radius $2\arccos(\lim_n |a_n|)$ [2] (see also [26, Chapter 4] for an approach using the five-diagonal representation $C(\boldsymbol{a})$). It is of interest to extend these results to a bigger class than the López one. With our techniques we can get information about $\{\operatorname{supp}\mu\}'$ when only one of the two López conditions is satisfied, or, even, under the more general condition $\lim_n |a_{n+1}/a_n| = 1$. Among other results, we prove that, if $\lim_n |a_{n+1}/a_n| = 1$, $\{\operatorname{supp}\mu\}'$ lies inside the union of closed arcs with centre at the limit points of the sequence $(-a_{n+1}/a_n)_{n\geqslant 1}$ and angular radius $2\arccos(\underline{\lim}_n |a_n|)$, which is a natural generalization of the result for the López class.

Section 4 is devoted to the study of the approximation of $\operatorname{supp}\mu$ by means of the spectra of the unitary truncations of $C(\boldsymbol{a})$ on ℓ_n^2, which means the approximation of the spectrum of an infinite unitary matrix by the spectra of finite unitary matrices. The proofs now include methods, not only from operator theory, but also from the theory of analytical functions. We prove that, for any measure μ on $\mathbb{T}$, there exist infinitely many sequences of unitary truncations whose spectra exactly converge to $\operatorname{supp}\mu$, in a strong sense that we will specify later on. This result proves a conjecture formulated by L. Golinskii in [14], concerning the coincidence of the support of a measure on $\mathbb{T}$ and the strong limit points of the zeros of the related para-orthogonal polynomials. We also present some other results that deal with weaker notions of convergence of the finite spectra, which are of interest in the following section.

Finally, in Section 5, we consider an application of the previous results, that is, the study of the convergence of rational approximants of the Carathéodory function of a measure on $\mathbb{T}$. It is known that the standard rational approximants constructed with the related orthogonal polynomials, or their reversed ones, converge on $\mathbb{C}\backslash\overline{\mathbb{D}}$ and $\mathbb{D}$, respectively. We focus our analysis on the study of the rational approximants related to the unitary truncations of $C(\boldsymbol{a})$ on ℓ^2_n, that always converge on $\mathbb{C}\backslash\mathbb{T}$, and are just the rational approximants constructed with the para-orthogonal polynomials [20]. The domain of convergence of these approximants is closely related to the asymptotic behaviour of the finite spectra of the above unitary truncations. Therefore, the results of the previous sections give information about the convergence of these approximants on $\mathbb{T}$, where the situation is more delicate. Some results in this direction for the standard rational approximants can be found in [21].

2. Normal truncations of $C(\boldsymbol{a})$

In what follows, given a Hilbert space H, $(\cdot,\cdot)$ is the corresponding inner product and $\|\cdot\|$ the related norm. We will deal with the set $\mathfrak{B}(H)$ of bounded linear operators on H. $\|\cdot\|$ also denotes the standard operator norm in $\mathfrak{B}(H)$ while, for any operator T on H,

$$\|T\|_S := \sup_{x\in S\backslash\{0\}} \frac{\|Tx\|}{\|x\|}, \quad \gamma(T;S) := \inf_{x\in S\backslash\{0\}} \frac{\|Tx\|}{\|x\|}, \quad \forall S\subset H,$$

and $\gamma(T) := \gamma(T;H)$. Given a sequence $(T_n)_{n\geqslant 1}$ of operators on H, $T_n \to T$ means that $\lim_n \|T_nx - Tx\| = 0$, $\forall x\in H$ (T is the strong limit of $(T_n)_{n\geqslant 1}$).

Let S be a subspace of H. If an operator T leaves S invariant, the operator on S defined by the restriction of T to S is denoted by $T_{\restriction S}$. In particular, $0_{\restriction S}$ and $1_{\restriction S}$ are the null and identity operators on S, respectively (the identity operator will be omitted when it is clear from the context). Also, if T is an operator on S, we define an operator on H by $\hat{T} := T\oplus 0_{\restriction S^\perp}$.

If $T\in\mathfrak{B}(H)$, $\sigma(T)$ is its spectrum and $\sigma_e(T)$ its essential spectrum. When T is normal it is known that $\sigma(T) = \{z\in\mathbb{C} : \gamma(z-T) = 0\}$. In fact, in this case, denoting by $d(\cdot,\cdot)$ the distance between points and sets in $\mathbb{C}$, we have $\gamma(z-T) = d(z,\sigma(T))$ for any $z\in\mathbb{C}$.

Let $T\in\mathfrak{B}(H)$ and Q be a projection on $S\subset H$ along $S'\subset H$. The operator $T[Q] := QT_{\restriction S}$ is called the truncation of T associated with Q, or the truncation of T on S along S'. $T[Q]$ is finite (co-finite) when S has finite dimension (co-dimension). If Q is an orthogonal projection we say that $T[Q]$ is an orthogonal truncation. To compare the operator with its truncation, it is convenient to consider $\hat{T}[Q] = T[Q]\oplus 0_{\restriction S^\perp} = QTP$, where P is the orthogonal projection on S. Notice that, if $T[Q]$ is bounded (for example, this is the case of a finite truncation), $\|\hat{T}[Q]\| = \|T[Q]\|$. Also, $\hat{T}[Q]^* = T[Q]^*\oplus 0_{\restriction S^\perp}$, so, any orthogonal truncation of a self-adjoint operator is self-adjoint too. However, in general, to get normal truncations of a normal operator can require non-orthogonal truncations.

In what follows, any infinite bounded matrix M is identified with the operator $T\in\mathfrak{B}(\ell^2)$ defined by $Tx = Mx$, $\forall x\in\ell^2$. T is called a band operator if M is a band matrix. The following result shows the interest in finding normal finite and co-finite truncations of a normal band operator.

Proposition 2.1. *Let $T\in\mathfrak{B}(\ell^2)$ be a normal band operator.*

1. *If T_n is a normal truncation of T on ℓ^2_n for $n\geqslant 1$ and $(\|T_n\|)_{n\geqslant 1}$ is bounded, then $\hat{T}_n \to T$ and*

$$\sigma(T) \subset \left\{z\in\mathbb{C} : \lim_n d(z,\sigma(T_n)) = 0\right\}.$$

2. *For any bounded normal truncation T_n of T on $\ell_n^{2\perp}$,*

$$\sigma_e(T_n) = \sigma_e(T).$$

Proof. Let T_n be a normal truncation of T on ℓ_n^2. If T is a $2N+1$-band operator, $T\ell_n^2 \subset \ell_{n+N}^2$ for $n \geqslant 1$. Therefore, $\hat{T}_n x = Tx$ if $x \in \ell_{n-N}^2$, $n > N$, and we get

$$\|\hat{T}_n x - Tx\| \leqslant (\|T_n\| + \|T\|)\|x - P_{n-N}x\|, \quad \forall n > N, \quad \forall x \in \ell^2.$$

Since $P_n \to 1$ and $(\|T_n\|)_{n\geqslant 1}$ is bounded we find that $\hat{T}_n \to T$.

If $\overline{\lim}_n d(z, \sigma(T_n)) > 0$, there exist $\delta > 0$ and an infinite set $\mathcal{I} \subset \mathbb{N}$ such that $d(z, \sigma(T_n)) \geqslant \delta$, $\forall n \in \mathcal{I}$. Since T_n is normal, $\gamma(z - T_n) \geqslant \delta$, $\forall n \in \mathcal{I}$. Hence, if P_n is the orthogonal projection on ℓ_n^2,

$$\|(zP_n - \hat{T}_n)x\| = \|(z - T_n)P_n x\| \geqslant \delta \|P_n x\|, \quad \forall x \in \ell^2, \quad \forall n \in \mathcal{I}. \tag{5}$$

Taking limits in (5) we obtain $\|(z - T)x\| \geqslant \delta\|x\|$, $\forall x \in \ell^2$, which, taking into account that T is normal, implies that $z \notin \sigma(T)$. This proves 1.

As for the second statement, let T_n be a normal truncation of T on $\ell_n^{2\perp}$. If T is $2N+1$-band, $T\ell_n^{2\perp} \subset \ell_{n-N}^{2\perp}$ for $n > N$. Hence, if $n \geqslant 1$, $\hat{T}_n x = Tx$ for $x \in \ell_{n+N}^{2\perp}$, and, thus, rank $(T - \hat{T}_n) \leqslant n + N$. Since $T - \hat{T}_n$ has finite rank, Weyl's theorem implies that $\sigma_e(T) = \sigma_e(\hat{T}_n) = \sigma_e(T_n)$. □

The first statement of the above proposition is the first step in establishing a numerical method for the approximation of the spectrum of an infinite bounded normal band operator. This statement will be improved in the case of finite normal truncations of $C(\boldsymbol{a})$, getting an equality instead of an inclusion (see Section 4), which can be used for the numerical approximation of the support of the related measure on $\mathbb{T}$.

The importance of the second assertion of Proposition 2.1 is that it can be used to extract properties of the essential spectrum of an infinite bounded normal band operator from the asymptotic behaviour of the coefficients of its diagonals. When applying Proposition 2.1 to co-finite truncations of $C(\boldsymbol{a})$, we can obtain properties of the derived set of the support of a measure on $\mathbb{T}$ from the asymptotic behaviour of the related Schur parameters (see Section 3).

Our next step is to study the normal truncations of $C(\boldsymbol{a})$ for an arbitrary sequence $\boldsymbol{a}$ in $\mathbb{D}$. This is equivalent to studying the normal truncations of U^μ, where μ is the measure on $\mathbb{T}$ related to $\boldsymbol{a}$. Concerning this problem we have the following result.

Theorem 2.2. *Let μ be a measure on $\mathbb{T}$ with infinite support and $\mathbb{P}_{m,n} :=$ span$\{z^m, z^{m+1}, \ldots, z^{m+n-1}\}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$. The normal truncations of U^μ on $\mathbb{P}_{m,n}$ ($\mathbb{P}_{m,n}^\perp$) are unitary, and they are parameterized by the points in $\mathbb{T}$. The normal truncation on $\mathbb{P}_{m,n}$ ($\mathbb{P}_{m,n}^\perp$) corresponding to a parameter $u \in \mathbb{T}$ is $U_{m,n}^\mu(u) := U^\mu[Q_{m,n}^\mu(u)]$, where $Q_{m,n}^\mu(u)$ is the projection on $\mathbb{P}_{m,n}$ ($\mathbb{P}_{m,n}^\perp$) along* span$\{z^m p_n^u\} \oplus \mathbb{P}_{m,n+1}^\perp$ (span$\{z^m q_n^u\} \oplus \mathbb{P}_{m+1,n-1}$), *and*

$$p_n^u(z) := z\varphi_{n-1}(z) + u\varphi_{n-1}^*(z), \quad q_n^u(z) := \varphi_n^*(z) - \overline{u}\varphi_n(z),$$

$(\varphi_n)_{n\geqslant 0}$ being the orthonormal polynomials in L^2_μ. If $(a_n)_{n\geqslant 1}$ are the Schur parameters of μ,

$$\|Q^\mu_{m,n}(u)\| = \sqrt{1+|u-a_n|^2/\rho_n^2}.$$

The spectrum of the truncation $U^\mu_{m,n}(u)$ on $\mathbb{P}_{m,n}$ is simple and coincides with the zeros of p^u_n.

Proof. The problem can be reduced to the study of the normal truncations $U^\mu_n = U^\mu[Q^\mu_n]$ of U^μ on $\mathbb{P}_n := \mathbb{P}_{0,n}$ ($\mathbb{P}_n^\perp$), since the normal truncations $U^\mu_{m,n} = U^\mu[Q^\mu_{m,n}]$ on $\mathbb{P}_{m,n}$ ($\mathbb{P}^\perp_{m,n}$) are related to the previous ones by $Q^\mu_{m,n} = z^m Q^\mu_n z^{-m}$ and $U^\mu_{m,n} = z^m U^\mu_n z^{-m}$.

Let U^μ_n be a truncation of U^μ on $\mathbb{P}_n$. For any $f\in\mathbb{P}_n$, the decomposition $U^\mu_n f = z(f-(f,\varphi_{n-1})\varphi_{n-1}) + (f,\varphi_{n-1})U^\mu_n\varphi_{n-1}$ gives

$$U^\mu_n f = zf-(f,\varphi_{n-1})p_n, \quad p_n = z\varphi_{n-1}-f_n, \quad f_n = U^\mu_n\varphi_{n-1}. \tag{6}$$

Thus, $U^\mu_n = U^\mu[Q^\mu_n]$, Q^μ_n being the projection on $\mathbb{P}_n$ along $\operatorname{span}\{p_n\}\oplus\mathbb{P}^\perp_{n+1}$.

From (6), for an arbitrary $f\in\mathbb{P}_n$, we get

$$U^{\mu*}_n f = z^{-1}(f-(f,\varphi^*_n)\varphi^*_n) - (f,p_n)\varphi_{n-1},$$

and, therefore,

$$U^\mu_n U^{\mu*}_n f = f-(f,\varphi^*_n)\varphi^*_n-(f,z\varphi_{n-1})z\varphi_{n-1}+(f,f_n)f_n, \tag{7}$$

$$\begin{aligned} U^{\mu*}_n U^\mu_n f = {} & f+(zf,f_n)\varphi_{n-1} \\ & +(f,\varphi_{n-1})\{z^{-1}(f_n-(f_n,\varphi^*_n)\varphi^*_n)+(\|p_n\|^2-2)\varphi_{n-1}\}. \end{aligned} \tag{8}$$

Let us suppose that U^μ_n is normal, that is $(U^\mu_n U^{\mu*}_n - U^{\mu*}_n U^\mu_n)f = 0$ for any $f\in\mathbb{P}_n$. Using (7) and (8) we find that

$$(f,f_n)f_n = (zf,f_n)\varphi_{n-1}, \quad \forall f\in z\mathbb{P}_{n-2}.$$

If $f\in z\mathbb{P}_{n-2}$ is such that $(f,f_n)\neq 0$, the above equality implies that f_n is proportional to φ_{n-1}, which gives a contradiction since $\varphi_{n-1}\perp z\mathbb{P}_{n-2}$. So, $(f,f_n)=(zf,f_n)=0$ for any $f\in z\mathbb{P}_{n-2}$, that is,

$$f_n\in(z\mathbb{P}_{n-2}+z^2\mathbb{P}_{n-2})^{\perp\mathbb{P}_n} = z\mathbb{P}^{\perp\mathbb{P}_n}_{n-1} = \operatorname{span}\{\varphi^*_{n-1}\}.$$

Therefore, $f_n = -u\varphi^*_{n-1}$, $u\in\mathbb{C}$. Then, if we take $f=1$ in (7) and (8), the condition $(U^\mu_n U^{\mu*}_n - U^{\mu*}_n U^\mu_n)1 = 0$ gives $|u|^2\varphi^*_{n-1} = \rho_n\varphi^*_n - \overline{a}_n z\varphi_{n-1}$, and the reversed form of (3) shows that $u\in\mathbb{T}$.

Moreover, if $f_n = -u\varphi^*_{n-1}$, $u\in\mathbb{T}$, we get from (7)

$$U^\mu_n U^{\mu*}_n f - f = (f,\varphi^*_{n-1})(\varphi^*_{n-1}-\rho_n\varphi^*_n+\overline{a}_n z\varphi_{n-1}) = 0,$$

and, hence, the finite truncation U^μ_n is unitary.

Let us consider now a truncation U^μ_n of U^μ on $\mathbb{P}^\perp_n$. The orthogonal decomposition $\mathbb{P}^\perp_n = z^{-1}\mathbb{P}^\perp_{n+1}\oplus\operatorname{span}\{z^{-1}\varphi^*_n\}$ gives, for any $f\in\mathbb{P}^\perp_n$, the equality $U^\mu_n f = z(f-(f,z^{-1}\varphi^*_n)z^{-1}\varphi^*_n) + (f,z^{-1}\varphi^*_n)U^\mu_n z^{-1}\varphi^*_n$, and, thus,

$$U^\mu_n f = zf-(zf,\varphi^*_n)q_n, \quad q_n = \varphi^*_n - g_n, \quad g_n = U^\mu_n z^{-1}\varphi^*_n. \tag{9}$$

So, $U^\mu_n = U^\mu[Q^\mu_n]$, Q^μ_n being the projection on $\mathbb{P}^\perp_n$ along $\operatorname{span}\{q_n\}\oplus z\mathbb{P}_{n-1}$.

From (9) we find that, for an arbitrary $f \in \mathbb{P}_n^{\perp}$,

$$U_n^{\mu *} f = z^{-1}(f - (f, z\varphi_{n-1})z\varphi_{n-1}) - (f, q_n)z^{-1}\varphi_n^*,$$

and

$$U_n^{\mu} U_n^{\mu *} f = f - (f, \varphi_n^*)\varphi_n^* - (f, z\varphi_{n-1})z\varphi_{n-1} + (f, g_n)g_n,$$

$$\begin{aligned} U_n^{\mu *} U_n^{\mu} f = f &+ (zf, g_n)z^{-1}\varphi_n^* \\ &+ (zf, \varphi_n^*)\{z^{-1}(g_n - (g_n, z\varphi_{n-1})z\varphi_{n-1}) + (\|q_n\|^2 - 2)z^{-1}\varphi_n^*\}. \end{aligned}$$

When $(U_n^{\mu} U_n^{\mu *} - U_n^{\mu *} U_n^{\mu})f = 0$ for any $f \in \mathbb{P}_n^{\perp}$, we find that

$$(f, g_n)g_n = (zf, g_n)z^{-1}\varphi_n^*, \quad \forall f \in z^{-1}\mathbb{P}_{n+2}^{\perp},$$

and, in a similar way to the previous case, we obtain that

$$g_n \in \mathbb{P}_n^{\perp(\mathbb{P}_{n+2} \cap z^{-1}\mathbb{P}_{n+2})} = \mathbb{P}_n^{\perp \mathbb{P}_{n+1}} = \text{span}\{\varphi_n\}.$$

Therefore, $g_n = \overline{u}\varphi_n$, $u \in \mathbb{C}$, and $(U_n^{\mu} U_n^{\mu *} - U_n^{\mu *} U_n^{\mu})z^{-1}\varphi_{n+1} = 0$ gives $|u|^2\varphi_n = \rho_n z\varphi_{n-1} + a_n\varphi_n^*$. From (4) we conclude that $u \in \mathbb{T}$.

Moreover, if $g_n = \overline{u}\varphi_n$, $u \in \mathbb{T}$,

$$\begin{aligned} U_n^{\mu} U_n^{\mu *} f - f &= (f, \varphi_n)(\varphi_n - \rho_n z\varphi_{n-1} - a_n\varphi_n^*) = 0, \\ U_n^{\mu *} U_n^{\mu} f - f &= u\rho_n(f, \varphi_{n-1})z^{-1}\varphi_n^* = 0, \end{aligned}$$

which shows that U_n^{μ} is unitary.

Notice that $p_n = \rho_n\varphi_n + (u - a_n)\varphi_{n-1}^*$ and $q_n = \rho_n\varphi_{n-1}^* + (\overline{a}_n - \overline{u})\varphi_n$. Therefore,

$$Q_n^{\mu}\varphi_{n-1}^* = \alpha_n\varphi_n, \quad Q_n^{\mu}\varphi_n = \beta_n\varphi_{n-1}^*,$$

where $\alpha_n = 1$, $\beta_n = (a_n - u)/\rho_n$ in the finite case and $\alpha_n = (\overline{u} - \overline{a}_n)/\rho_n$, $\beta_n = 1$ in the co-finite case. In any of the two cases, $Q^{\mu}_{n \restriction z\mathbb{P}_{n-1}}$ and $Q^{\mu}_{n \restriction \mathbb{P}_{n+1}^{\perp}}$ are the unit or null operators. Since $L_{\mu}^2 = z\mathbb{P}_{n-1} \oplus \text{span}\{\varphi_{n-1}^*, \varphi_n\} \oplus \mathbb{P}_{n+1}^{\perp}$ we find that $\|Q_n^{\mu}\| = \|Q^{\mu}_{n \restriction \text{span}\{\varphi_{n-1}^*, \varphi_n\}}\| = \sqrt{1 + |u - a_n|^2/\rho_n^2}$.

Finally, 1 is a cyclic vector for any finite normal truncation U_n^{μ} since $\text{span}\{U_n^{\mu k} 1\}_{k=0}^{n-1} = \mathbb{P}_n$, so, the spectrum of U_n^{μ} is simple. The identity (6) implies that any eigenvalue of U_n^{μ} is a zero of p_n, hence, the n different eigenvalues of U_n^{μ} must fulfill the n (simple) zeros of p_n. □

Remark 2.3. The polynomials p_n^u and q_n^u can be understood as the substitutes for $\rho_n\varphi_n$ and $\rho_n\varphi_{n-1}^*$, when changing $a_n \in \mathbb{D}$ by $u \in \mathbb{T}$ in the nth step of (3) and the reversed version of (4), respectively. In fact, using (3) and (4) we see that these polynomials are related by

$$p_n^u = \frac{u - a_n}{\rho_n} q_n^v, \quad q_n^v = \frac{\overline{a}_n - \overline{v}}{\rho_n} p_n^u, \quad u = -v\,\frac{1 - a_n\overline{v}}{1 - \overline{a}_n v}, \quad v = -u\,\frac{1 - a_n\overline{u}}{1 - \overline{a}_n u}. \tag{10}$$

They are called para-orthogonal polynomials of order n associated with the measure μ [20]. It was known that they have simple zeros lying on $\mathbb{T}$, which is in agreement with Theorem 2.2. Notice that the freedom in the parameter $u \in \mathbb{T}$ means that we can arbitrarily fix in $\mathbb{T}$ one of the zeros of a para-orthogonal polynomial with given order, the rest of the zeros being determined by this choice.

The importance of Theorem 2.2 is that it provides and, at the same time, closes the possible ways of applying Proposition 2.1 to the unitary matrix $C(\boldsymbol{a})$. We will identify any truncation of $C(\boldsymbol{a})$ on ℓ^2_n ($\ell^{2\perp}_n$) with its matrix representation with respect to $\{e_k\}_{k\leqslant n}$ ($\{e_k\}_{k>n}$). The matrix form of the normal truncations of $C(\boldsymbol{a})$ can be obtained using the decomposition

$$C(\boldsymbol{a}) = C(a_1, \ldots, a_n) \oplus C(\boldsymbol{a}^{(n)}; a_n), \quad |a_n| = 1, \quad \boldsymbol{a}^{(n)} := (a_{n+j})_{j\geqslant 1}, \tag{11}$$

where $C(a_1, \ldots, a_n)$ is the principal matrix of $C(\boldsymbol{a})$ of order n, and

$$C(\boldsymbol{a}; u) := \begin{cases} WC(\overline{u}\boldsymbol{a})W^* & \text{even } k, \\ (WC(\overline{u}\boldsymbol{a})W^*)^t & \text{odd } k, \end{cases} \quad W := \begin{pmatrix} w & & & & \\ & \overline{w} & & & \\ & & w & & \\ & & & \overline{w} & \\ & & & & \ddots \end{pmatrix} \tag{12}$$

with $u = w^2$. The factorization $C(\boldsymbol{a}) = C_{\rm o}(\boldsymbol{a})C_{\rm e}(\boldsymbol{a})$ is useful, where

$$C_{\rm o}(\boldsymbol{a}) := \begin{pmatrix} \Theta(a_1) & & & \\ & \Theta(a_3) & & \\ & & \Theta(a_5) & \\ & & & \ddots \end{pmatrix},$$

$$C_{\rm e}(\boldsymbol{a}) := \begin{pmatrix} 1 & & & \\ & \Theta(a_2) & & \\ & & \Theta(a_4) & \\ & & & \ddots \end{pmatrix}, \tag{13}$$

$$\Theta(a) := \begin{pmatrix} -a & \rho \\ \rho & \overline{a} \end{pmatrix}, \quad \rho := \sqrt{1-|a|^2}, \quad |a| \leqslant 1.$$

Also, $C(a_1, \ldots, a_n) = C_{\rm o}(a_1, \ldots, a_n)C_{\rm e}(a_1, \ldots, a_n)$, where $C_{\rm o}(a_1, \ldots, a_n)$ and $C_{\rm e}(a_1, \ldots, a_n)$ are the principal submatrices of order n of $C_{\rm o}(\boldsymbol{a})$ and $C_{\rm e}(\boldsymbol{a})$, respectively. All these properties hold for $|a_n| \leqslant 1$ [8].

If $(\chi_n)_{n\geqslant 0}$ are the Laurent orthonormal polynomials related to a sequence $\boldsymbol{a}$ of Schur parameters, we also have (see [7])

$$\begin{cases} (\chi_{n-1}(z) \quad \chi_n(z)) = (\chi_{n-1*}(z) \quad \chi_{n*}(z))\,\Theta(a_n) & \text{even } n, \\ z\,(\chi_{n-1*}(z) \quad \chi_{n*}(z)) = (\chi_{n-1}(z) \quad \chi_n(z))\,\Theta(a_n) & \text{odd } n. \end{cases} \tag{14}$$

Corollary 2.4. *For any sequence* $\boldsymbol{a}$ *in* $\mathbb{D}$, *the normal truncations of* $C(\boldsymbol{a})$ *on* ℓ^2_n *and* $\ell^{2\perp}_n$ *are unitary and they have, respectively, the form* $C(a_1, \ldots, a_{n-1}, u)$ *and* $C(\boldsymbol{a}^{(n)}; u)$, *with* $u \in \mathbb{T}$. *For both kinds of truncations, the related projections* $Q_n(\boldsymbol{a}; u)$ *satisfy* $\|Q_n(\boldsymbol{a}; u)\| = \sqrt{1 + |u - a_n|^2/\rho_n^2}$. *For any* $u \in \mathbb{T}$ *the spectrum of* $C(a_1, \ldots, a_{n-1}, u)$ *is simple and coincides with the zeros of the para-orthogonal polynomial* p_n^u *associated with the measure related to* $\boldsymbol{a}$.

Proof. Let μ be the measure on $\mathbb{T}$ whose sequence of Schur parameters is $\boldsymbol{a}$, and let $(\chi_n)_{n\geqslant 0}$ be the corresponding Laurent orthonormal polynomials. Since $\operatorname{span}\{\chi_0, \ldots, \chi_{n-1}\} = \mathbb{P}_{m,n}$, $m =$

$-[\frac{n-1}{2}]$, the unitary equivalence between U^μ and $C(\boldsymbol{a})$ implies that the normal truncations of $C(\boldsymbol{a})$ on ℓ^2_n ($\ell^{2\perp}_n$) are given by the matrix representations of the normal truncations of U^μ on $\mathbb{P}_{m,n}$ ($\mathbb{P}^\perp_{m,n}$), when using the basis $\{\chi_j\}_{j<n}$ ($\{\chi_j\}_{j\geqslant n}$). So, it just remains to prove that these representations have the matrix form given in the corollary.

Let $\hat{\boldsymbol{a}}$ be the sequence obtained from $\boldsymbol{a}$ when substituting a_n by $u \in \mathbb{T}$. Property (11) implies that $C(\hat{\boldsymbol{a}}) = C(a_1, \ldots, a_{n-1}, u) \oplus C(\boldsymbol{a}^{(n)}; u)$. If $\Delta := C(\boldsymbol{a}) - C(\hat{\boldsymbol{a}})$ and $\boldsymbol{\chi} := (\chi_n)_{n\geqslant 0}$, (13) and (14) lead to

$$\Delta^t \boldsymbol{\chi}(z) = z^m (b_n p^u_n(z) + d_n q^u_n(z)), \quad b_n \in \ell^2_n, \quad d_n \in \ell^{2\perp}_n.$$

Since $C(\boldsymbol{a})^t \boldsymbol{\chi}(z) = z\boldsymbol{\chi}(z)$, if $\boldsymbol{\chi}_n := (\chi_j)_{j<n}$ and $\boldsymbol{\chi}^{(n)} := (\chi_j)_{j\geqslant n}$,

$$z\boldsymbol{\chi}_n(z) = C(a_1, \ldots, a_{n-1}, u)^t \boldsymbol{\chi}_n(z) + \beta_n z^m p^u_n(z), \quad \beta_n \in \mathbb{C}^n,$$
$$z\boldsymbol{\chi}^{(n)}(z) = C(\boldsymbol{a}^{(n)}; u)^t \boldsymbol{\chi}^{(n)}(z) + \delta_n z^m q^u_n(z), \quad \delta_n \in \ell^2.$$

This shows that $C(a_1, \ldots, a_{n-1}, u)$ ($C(\boldsymbol{a}^{(n)}; u)$) is the matrix representation of the normal truncation of U^μ on $\mathbb{P}_{m,n}$ ($\mathbb{P}^\perp_{m,n}$) along $\operatorname{span}\{z^m p^u_n\} \oplus \mathbb{P}^\perp_{m,n+1}$ ($\operatorname{span}\{z^m q^u_n\} \oplus \mathbb{P}_{m+1,n-1}$) with respect to $\boldsymbol{\chi}_n$ ($\boldsymbol{\chi}^{(n)}$). □

Notice that the normal truncations provided by Theorem 2.2 and Corollary 2.4 are always non-orthogonal. These truncations have a remarkable meaning. Concerning the finite ones, the spectrum of the matrices $C(a_1, \ldots, a_n, u)$, $u \in \mathbb{T}$, provides the nodes of the Szegő quadrature formulas [20] for the measure μ corresponding to $C(\boldsymbol{a})$. In fact, these matrices are the five-diagonal representations of the unitary multiplication operators related to the finitely supported measures $\mu_{u,n}$ associated with such quadrature formulas [8]. As for the co-finite truncations, for any $u \in \mathbb{T}$, the matrices $C(\boldsymbol{a}^{(n)}; u)$ are unitarily equivalent to $C(\overline{u}\boldsymbol{a}^{(n)})$, which are the five-diagonal representations corresponding to the family $\mu^{(n)}_u$ of Aleksandrov measures related to the n-associated orthogonal polynomials.

When applied to the truncations of $C(\boldsymbol{a})$ given in the previous corollary, Proposition 2.1 states that $\{\operatorname{supp} \mu\}' = \{\operatorname{supp} \mu^{(n)}_u\}'$ for any $u \in \mathbb{T}$, and $\operatorname{supp} \mu \subset \{z \in \mathbb{C} : \lim_n d(z, \operatorname{supp} \mu_{u_n,n}) = 0\}$ for any sequence $\boldsymbol{u} = (u_n)_{n\geqslant 1}$ in $\mathbb{T}$. These relations were previously known (see [14, Theorem 8] for the result concerning $\mu_{u,n}$ and [23,16,26,27] for results related to $\mu^{(n)}_u$). The relevance of our approach is that it proves that the only possibility of applying Proposition 2.1 to $C(\boldsymbol{a})$ necessarily leads to the measures $\mu_{u,n}$ and $\mu^{(n)}_u$.

3. Co-finite truncations of $C(\boldsymbol{a})$ and the derived set of the support of the measure

If $\boldsymbol{a}$ is the sequence of Schur parameters for the measure μ on $\mathbb{T}$, our aim is to relate $\{\operatorname{supp} \mu\}' = \sigma_{\mathrm{e}}(C(\boldsymbol{a}))$ and the asymptotic behaviour of $\boldsymbol{a}$ using Proposition 2.1.2 and some results of operator theory. Concerning the notation, if $T \in \mathfrak{B}(H)$ is normal, we denote by E_T its spectral measure, so that

$$(Tx, y) = \int_{\mathbb{C}} \lambda d(E_T(\lambda)x, y), \quad \forall x, y \in H.$$

In fact, the above expression can be understood as an integral over $\sigma(T)$.

As for the results of operator theory that we will apply, we start with a characterization of the essential spectrum of a normal operator and a lower bound for the distance from a point to this essential spectrum. In what follows we use the notation $D_\varepsilon(z)$ for an open disk of centre $z \in \mathbb{C}$ and radius $\varepsilon > 0$.

Proposition 3.1. *Let $T \in \mathfrak{B}(H)$ be normal. A point $z \in \mathbb{C}$ lies on $\sigma_e(T)$ if and only if $\gamma(z - T; S) = 0$ for any subspace $S \subset H$ with finite co-dimension. Moreover, given an arbitrary subspace $S \subset H$ with finite co-dimension, $d(z, \sigma_e(T)) \geqslant \gamma(z - T; S)$ for any $z \in \mathbb{C}$.*

Proof. A point $z \in \mathbb{C}$ belongs to $\sigma_e(T)$ if and only if rank $E_T(D_\varepsilon(z))$ is infinite for any $\varepsilon > 0$. Therefore, if $z \notin \sigma_e(T)$, $S_\varepsilon := E_T(\mathbb{C}\backslash D_\varepsilon(z))H$ is a subspace with finite co-dimension for some $\varepsilon > 0$. Moreover,

$$\|(z - T)x\|^2 = \int_{\mathbb{C}\backslash D_\varepsilon(z)} |z - \lambda|^2 \, d(E_T(\lambda)x, x) \geqslant \varepsilon^2 \|x\|^2, \quad \forall x \in S_\varepsilon,$$

which proves that $\gamma(z - T; S_\varepsilon) \geqslant \varepsilon > 0$.

If, on the contrary, $z \in \sigma_e(T)$, $S'_\varepsilon := E_T(D_\varepsilon(z))H$ has infinite dimension for any $\varepsilon > 0$ and, thus, given a subspace $S \subset H$ with finite co-dimension there always exists a non-null vector $x_\varepsilon \in S \cap S'_\varepsilon$. Then,

$$\|(z - T)x_\varepsilon\|^2 = \int_{D_\varepsilon(z)} |z - \lambda|^2 \, d(E_T(\lambda)x_\varepsilon, x_\varepsilon) \leqslant \varepsilon^2 \|x_\varepsilon\|^2.$$

Since ε is arbitrary, we conclude that $\gamma(z - T; S) = 0$. Moreover, given $z \in \mathbb{C}$ and a subspace $S \subset H$ with finite co-dimension,

$$\gamma(z - T; S) \leqslant |z - w| + \gamma(w - T; S) = |z - w|, \quad \forall w \in \sigma_e(T),$$

which proves that $d(z, \sigma_e(T)) \geqslant \gamma(z - T; S)$. □

Concerning perturbative results, if $T_0, T \in \mathfrak{B}(H)$ and T_0 is normal, it is known that $\sigma(T) \subset \{z \in \mathbb{C} : d(z, \sigma(T_0)) \leqslant \|T - T_0\|\}$. The next result is the analogue for the essential spectrum of normal operators.

Proposition 3.2. *If $T_0, T \in \mathfrak{B}(H)$ are normal, for any subspace $S \subset H$ with finite co-dimension*

$$\sigma_e(T) \subset \{z \in \mathbb{C} : d(z, \sigma_e(T_0)) \leqslant \|T - T_0\|_S\}.$$

Proof. Suppose that $d(z, \sigma_e(T_0)) > \|T - T_0\|_S$. Consider a real number ε such that $d(z, \sigma_e(T_0)) > \varepsilon > \|T - T_0\|_S$. The subspace $S_\varepsilon := E_{T_0}(\mathbb{C}\backslash D_\varepsilon(z))H$ has finite co-dimension and, similarly to the proof of the previous proposition, $\gamma(z - T_0; S_\varepsilon) \geqslant \varepsilon$. $S'_\varepsilon := S \cap S_\varepsilon$ has also finite co-dimension and

$$\gamma(z - T; S'_\varepsilon) \geqslant \gamma(z - T_0; S'_\varepsilon) - \|(T - T_0)\|_{S'_\varepsilon} \geqslant \gamma(z - T_0; S_\varepsilon) - \|(T - T_0)\|_S.$$

Therefore, $\gamma(z - T; S'_\varepsilon) \geqslant \varepsilon - \|T - T_0\|_S > 0$ and, from Proposition 3.1, $z \notin \sigma_e(T)$. □

When applying to $C(\boldsymbol{a})$ and its co-finite normal truncations $C(\boldsymbol{a}^{(n)}; u)$, $u \in \mathbb{T}$, Propositions 2.1.2, 3.1 and 3.2 give the following result. For convenience, given an operator $T \in \mathfrak{B}(\ell^2)$, we use the notation $\|T\|_n := \|T\|_{\ell_n^{2\perp}}$, $\gamma_n(T) := \gamma(T; \ell_n^{2\perp})$, $n \geqslant 1$, and $\|T\|_0 := \|T\|$, $\gamma_0(T) := \gamma(T)$.

Theorem 3.3. *Let* $\boldsymbol{a}$ *be the sequence of Schur parameters of a measure* μ *on* $\mathbb{T}$, $\boldsymbol{b}$ *a sequence in* $\overline{\mathbb{D}}$, $u \in \mathbb{T}$, $m \geqslant 0$ *and* $z \in \mathbb{C}$.

1. $d(z, \{\operatorname{supp}\mu\}') \geqslant \sup_{n\geqslant 0} \gamma_m(z - C(\overline{u}\boldsymbol{a}^{(n)}))$.
2. $\inf_{n\geqslant 0} \|C(\overline{u}\boldsymbol{a}^{(n)}) - C(\boldsymbol{b}^{(n)})\|_m < d(z, \sigma_e(C(\boldsymbol{b}))) \;\Rightarrow\; z \notin \{\operatorname{supp}\mu\}'$.

Proof. From Proposition 2.1.2, $\sigma_e(C(\overline{u}\boldsymbol{a}^{(n)})) = \sigma_e(C(\boldsymbol{a})) = \{\operatorname{supp}\mu\}'$. So, a direct application of Proposition 3.1 gives $d(z, \{\operatorname{supp}\mu\}') \geqslant \gamma_m(z - C(\overline{u}\boldsymbol{a}^{(n)}))$ for any m, which proves the first statement.

Let us suppose now that $\inf_{n\geqslant 0} \|C(\overline{u}\boldsymbol{a}^{(n)}) - C(\boldsymbol{b}^{(n)})\|_m < d(z, \sigma_e(C(\boldsymbol{b})))$. Then, for some n, $\|C(\overline{u}\boldsymbol{a}^{(n)}) - C(\boldsymbol{b}^{(n)})\|_m < d(z, \sigma_e(C(\boldsymbol{b})))$. Since Proposition 2.1.2 implies that $\sigma_e(C(\boldsymbol{b})) = \sigma_e(C(\boldsymbol{b}^{(n)}))$, it follows from Proposition 3.2 that $z \notin \sigma_e(C(\overline{u}\boldsymbol{a}^{(n)}))$. Hence, using again Proposition 2.1.2, we find that $z \notin \sigma_e(C(\boldsymbol{a})) = \{\operatorname{supp}\mu\}'$. □

For the application of the preceding propositions we have to obtain lower bounds for $\gamma_m(C(\boldsymbol{a}) - C(\boldsymbol{b}))$ and upper bounds for $\|C(\boldsymbol{a}) - C(\boldsymbol{b})\|_m$, $\boldsymbol{a}$ and $\boldsymbol{b}$ being sequences in $\overline{\mathbb{D}}$. This is all we need since $C(\boldsymbol{b}) = z$ for $b_n = (-z)^n$ with $z \in \mathbb{T}$. Notice that

$$\left\| \begin{pmatrix} -\alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \left\| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\| \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|, \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2.$$

Hence, from the equality

$$C(\boldsymbol{a}) - C(\boldsymbol{b}) = (C_o(\boldsymbol{a}) - C_o(\boldsymbol{b}))\, C_e(\boldsymbol{a}) + C_o(\boldsymbol{b})(C_e(\boldsymbol{a}) - C_e(\boldsymbol{b})),$$

we get

$$\begin{aligned} \gamma_m(C(\boldsymbol{a}) - C(\boldsymbol{b})) &\geqslant \inf_{\substack{j\geqslant m-1 \\ \text{odd (even) } j}} k(a_j, b_j) - \sup_{\substack{j\geqslant m-1 \\ \text{even (odd) } j}} k(a_j, b_j), \\ \|C(\boldsymbol{a}) - C(\boldsymbol{b})\|_m &\leqslant \sup_{\substack{j\geqslant m-1 \\ \text{odd } j}} k(a_j, b_j) + \sup_{\substack{j\geqslant m-1 \\ \text{even } j}} k(a_j, b_j), \end{aligned} \tag{15}$$

where $k(x_1, x_2) := \gamma(\Theta(x_1) - \Theta(x_2)) = \|\Theta(x_1) - \Theta(x_2)\|$, that is,

$$k(x_1, x_2) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}, \quad |x_j| \leqslant 1, \quad y_j := \sqrt{1 - |x_j|^2}. \tag{16}$$

Equipped with these results, we can apply Theorem 3.3 in different ways to get information about the derived set of the support of a measure on $\mathbb{T}$ from the behaviour of the related Schur parameters. The first example of this is the next theorem. In what follows, given $z, w \in \mathbb{C}$, $w = e^{i\omega}z$, $\omega \in [0, 2\pi)$, we denote $(z, w) := \{e^{i\theta}z : \theta \in (0, \omega)\}$, $[z, w] := \{e^{i\theta}z : \theta \in [0, \omega]\}$. Also, for any $z \in \mathbb{C}$ and $\alpha \in [0, \pi]$, $\Delta_\alpha(z) := [e^{i\alpha}z, e^{-i\alpha}z]$, $\Gamma_\alpha(z) := (e^{-i\alpha}z, e^{i\alpha}z)$ and $\overline{\Gamma}_\alpha(z) := [e^{-i\alpha}z, e^{i\alpha}z]$. Besides, for any sequence $\boldsymbol{a}$ in $\mathbb{C}$ and any $z \in \mathbb{C}$ we define $\boldsymbol{a}(z) := (a_n(z))_{n\geqslant 1}$ by $a_n(z) := \overline{z}^n a_n$.

Theorem 3.4. *Let* μ *be a measure on* $\mathbb{T}$ *with a sequence* $\boldsymbol{a}$ *of Schur parameters. Assume that for some* $\lambda \in \mathbb{T}$ *the limit points of the odd and even subsequences of* $\boldsymbol{a}(-\lambda)$ *are separated by a band*

$\mathcal{B}(u,\alpha_1,\alpha_2) := \{z \in \mathbb{C} : \cos\alpha_2 < \mathrm{Re}(\overline{u}z) < \cos\alpha_1\}$, $u \in \mathbb{T}$, $0 \leqslant \alpha_1 < \alpha_2 \leqslant \pi$. *Then,*

$$\{\mathrm{supp}\,\mu\}' \subset \triangle_\alpha(\lambda), \quad \sin\frac{\alpha}{2} = \max\left\{\sin\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}, \cos\frac{\alpha_1}{2} - \cos\frac{\alpha_2}{2}\right\}.$$

Proof. Let $\boldsymbol{b}$ be defined by $b_n := (-\lambda)^n$, $\lambda \in \mathbb{T}$. $C(\boldsymbol{b}^{(n)})$ is diagonal with diagonal elements equal to λ, except the first one that is $(-1)^n\lambda^{n+1}$. Therefore, for $m \geqslant 1$, $\gamma_m(\lambda - C(\overline{u}\boldsymbol{a}^{(n)})) = \gamma_m(C(\boldsymbol{b}^{(n)}) - C(\overline{u}\boldsymbol{a}^{(n)}))$ and, from (15), we get

$$\sup_{n\geqslant 0} \gamma_m(\lambda - C(\overline{u}\boldsymbol{a}^{(n)})) \geqslant \sup_{n\geqslant 0} \inf_{\substack{\text{odd (even) } j \\ j\geqslant n+m-1}} k(\overline{u}a_j, b_j) - \inf_{n\geqslant 0} \sup_{\substack{\text{odd (even) } j \\ j\geqslant n+m-1}} k(\overline{u}a_j, b_j)$$
$$= \varliminf_{\text{odd (even) } n} \sqrt{2(1-\mathrm{Re}(\overline{u}a_n(-\lambda)))} - \varlimsup_{\text{even (odd) } n} \sqrt{2(1-\mathrm{Re}(\overline{u}a_n(-\lambda)))}.$$

Assume that the limit points of the even and odd subsequences of $\boldsymbol{a}(-\lambda)$ are separated by the band $\mathcal{B}(u,\alpha_1,\alpha_2)$. Then,

$$\varliminf_{\text{even (odd) } n} \mathrm{Re}(\overline{u}a_n(-\lambda)) \geqslant \cos\alpha_1, \qquad \varlimsup_{\text{odd (even) } n} \mathrm{Re}(\overline{u}a_n(-\lambda)) \leqslant \cos\alpha_2,$$

which gives

$$\sup_{n\geqslant 0} \gamma_m(\lambda - C(\overline{u}\boldsymbol{a}^{(n)})) \geqslant 2\sin\frac{\alpha_2}{2} - 2\sin\frac{\alpha_1}{2}.$$

Since the limit points of the even and odd subsequences of $\boldsymbol{a}(-\lambda)$ are separated by the band $\mathcal{B}(-u, \pi-\alpha_2, \pi-\alpha_1)$ too, we also get

$$\sup_{n\geqslant 0} \gamma_m(\lambda - C(-\overline{u}\boldsymbol{a}^{(n)})) \geqslant 2\cos\frac{\alpha_1}{2} - 2\cos\frac{\alpha_2}{2}.$$

So, if we define $\alpha \in (0,\pi]$ by $\sin\frac{\alpha}{2} = \max\{\sin\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}, \cos\frac{\alpha_1}{2} - \cos\frac{\alpha_2}{2}\}$, Proposition 3.3.1 gives $d(\lambda, \{\mathrm{supp}\,\mu\}') \geqslant 2\sin\frac{\alpha}{2}$, that is, $\{\mathrm{supp}\,\mu\}' \subset \triangle_\alpha(\lambda)$. □

Given $\lambda \in \mathbb{T}$, the above theorem ensures that $\lambda \notin \{\mathrm{supp}\,\mu\}'$ if the limit points of the odd and even subsequences of $\boldsymbol{a}(-\lambda)$ can be separated by a straight line. In particular, $\lambda \notin \{\mathrm{supp}\,\mu\}'$ if the limit points of $\boldsymbol{a}(\lambda)$ lie on an open half-plane whose boundary contains the origin, because then, the limit points of the odd and even subsequences of $\boldsymbol{a}(-\lambda)$ are separated by the straight line that limits such half-plane. In fact, we have the following immediate consequence of Theorem 3.4.

Corollary 3.5. *Let μ be a measure on $\mathbb{T}$ with a sequence $\boldsymbol{a}$ of Schur parameters. Assume that for some $\lambda \in \mathbb{T}$ the limit points of $\boldsymbol{a}(\lambda)$ lie on $\mathcal{D}(u,\alpha_0) := \{z \in \mathbb{C} : \mathrm{Re}(\overline{u}z) \geqslant \cos\alpha_0\}$, $u \in \mathbb{T}$, $0 \leqslant \alpha_0 < \pi/2$. Then,*

$$\{\mathrm{supp}\,\mu\}' \subset \triangle_\alpha(\lambda), \quad \cos\frac{\alpha}{2} = \sqrt{\sin\alpha_0}.$$

Proof. Under the assumptions of the corollary, the limit points of the odd and even subsequences of $\boldsymbol{a}(-\lambda)$ are separated by the band $\mathcal{B}(u,\alpha_0,\pi-\alpha_0)$. So, the direct application of

Theorem 3.4 proves that $\{\operatorname{supp}\mu\}' \subset \Delta_\alpha(\lambda)$ with $\alpha \in (0,\pi]$ given by $\sin\frac{\alpha}{2} = \cos\frac{\alpha_0}{2} - \sin\frac{\alpha_0}{2}$, that is, $\cos^2\frac{\alpha}{2} = \sin\alpha_0$. $\square$

Remark 3.6. If μ is the measure related to the sequence $\boldsymbol{a}$ of Schur parameters, the measure obtained by rotating μ through an angle θ is associated with the sequence $\boldsymbol{a}(e^{i\theta})$. Thus, Proposition 3.4 and Corollary 3.5 are just the rotated versions of the following basic statements:

1. If the limit points of the odd and even subsequences of $\boldsymbol{a}$ are separated by a band obtained by a rotation of $\cos\alpha_1 < \operatorname{Re}(z) < \cos\alpha_2$, then $\{\operatorname{supp}\mu\}' \subset \Delta_\alpha(-1)$ with $\sin\frac{\alpha}{2} = \max\{\sin\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}, \cos\frac{\alpha_1}{2} - \cos\frac{\alpha_2}{2}\}$.
2. If the limit points of $\boldsymbol{a}$ lie on a domain obtained by a rotation of $\operatorname{Re}(z) \geqslant \cos\alpha_0 > 0$, then $\{\operatorname{supp}\mu\}' \subset \Delta_\alpha(1)$ with $\cos\frac{\alpha}{2} = \sqrt{\sin\alpha_0}$.

Let us show an example of application of the previous results. In what follows we denote by $\mathfrak{L}\{\boldsymbol{a}\}$ the set of limit points of a sequence $\boldsymbol{a}$ in $\mathbb{C}$.

Example 3.7. $\mathfrak{L}\{\boldsymbol{a}(\lambda)\} = \{a, b\}$, $\lambda \in \mathbb{T}$, $a \neq b$, $a, b \neq 0$.

Let $\boldsymbol{a}$ be the Schur parameters of a measure μ on $\mathbb{T}$. With the help of the previous results we can get information about the case in which we just know that $\boldsymbol{a}(\lambda)$ has two different subsequential limit points a, b, no matter from which subsequence. Suppose that $0 < |a| \leqslant |b|$ and let $\frac{b-a}{|b-a|} = \frac{a}{|a|}e^{i\theta}$, $\theta \in (-\pi, \pi]$. Then, $\{a, b\} \subset \mathcal{D}(u, \alpha_0)$ where

$$u = \frac{a}{|a|}, \quad \cos\alpha_0 = |a|, \quad \text{if } |\theta| \leqslant \frac{\pi}{2},$$
$$u = -\operatorname{sign}(\theta) i\, \frac{b-a}{|b-a|}, \quad \cos\alpha_0 = |a|\sin|\theta|, \quad \text{if } |\theta| > \frac{\pi}{2}.$$

Therefore, using Corollary 3.5 we find that, if $\theta \neq \pi$ (which means that $\frac{b}{|b|} \neq -\frac{a}{|a|}$), $\{\operatorname{supp}\mu\}' \subset \Delta_\alpha(\lambda)$ with

$$\cos\frac{\alpha}{2} = \begin{cases} \sqrt[4]{1-|a|^2} & \text{if } |\theta| \leqslant \frac{\pi}{2}, \\ \sqrt[4]{1-|a|^2\sin^2\theta} & \text{if } |\theta| > \frac{\pi}{2}. \end{cases}$$

The next results use the second statement of Theorem 3.3. This requires the comparison of the matrix $C(\boldsymbol{a})$ related to a measure on $\mathbb{T}$ with another matrix $C(\boldsymbol{b})$ with known essential spectrum. The simplest case where the essential spectrum of $C(\boldsymbol{b})$ is known is when it is a diagonal matrix, which means that $\boldsymbol{b}$ is a sequence in $\mathbb{T}$. Applying Theorem 3.3.2 to the comparison between $C(\boldsymbol{a})$ and $C(\boldsymbol{b})$ with a suitable choice for $\boldsymbol{b}$ in $\mathbb{T}$, we get the following result.

Proposition 3.8. *Let $a_n = |a_n|u_n$ ($u_n \in \mathbb{T}$) be the Schur parameters of a measure μ on $\mathbb{T}$, and assume that $c(\boldsymbol{a}) := \min\{c_1(\boldsymbol{a}), c_2(\boldsymbol{a})\} < 1$, where*

$$c_1(\boldsymbol{a}) := \frac{1}{2}\,\overline{\lim_n}\left(\||a_{n+1}| - |a_n|\| + \rho_n + \rho_{n+1}\right),$$
$$c_2(\boldsymbol{a}) := \overline{\lim_{\text{odd}\, n}}\sqrt{\frac{1-|a_n|}{2}} + \overline{\lim_{\text{even}\, n}}\sqrt{\frac{1-|a_n|}{2}}.$$

Then,

$$\{\operatorname{supp}\mu\}' \subset \bigcup_{\lambda \in \mathfrak{L}\{\overline{u}_n u_{n+1}\}} \Delta_\alpha(\lambda), \quad \cos\frac{\alpha}{2} = c(\boldsymbol{a}).$$

Proof. Let us define the sequence $\boldsymbol{b}$ by $b_n := u_n$. Since $u_n \in \mathbb{T}$, $C(\boldsymbol{b})$ is the diagonal matrix

$$C(\boldsymbol{b}) = -\begin{pmatrix} u_1 & & & \\ & \overline{u}_1 u_2 & & \\ & & \overline{u}_2 u_3 & \\ & & & \ddots \end{pmatrix}, \tag{17}$$

and, hence, $\sigma_{\mathrm{e}}(C(\boldsymbol{b})) = -\mathfrak{L}\{\overline{u}_n u_{n+1}\}$. Using (15) we get

$$\inf_{n \geqslant 0} \|C(\boldsymbol{a}^{(n)}) - C(\boldsymbol{b}^{(n)})\| \leqslant 2c_2(\boldsymbol{a}).$$

We can find another upper bound for $\inf_{n\geqslant 0} \|C(\boldsymbol{a}^{(n)}) - C(\boldsymbol{b}^{(n)})\|$ in the following way. The factorizations $C(\boldsymbol{a}) - C(\boldsymbol{b}) = C_{\mathrm{o}}(\boldsymbol{a})(C_{\mathrm{e}}(\boldsymbol{a}) - C_{\mathrm{o}}^*(\boldsymbol{a})C(\boldsymbol{b}))$ and $C_{\mathrm{e}}(\boldsymbol{a}) - C_{\mathrm{o}}^*(\boldsymbol{a})C(\boldsymbol{b}) = A(\boldsymbol{a})B(\boldsymbol{b})$, where

$$A(\boldsymbol{a}) := \begin{pmatrix} 1-|a_1| & \rho_1\overline{u}_1 & & & & \\ \rho_1 u_1 & |a_1|-|a_2| & \rho_2 u_2 & & & \\ & \rho_2\overline{u}_2 & |a_2|-|a_3| & \rho_3\overline{u}_3 & & \\ & & \rho_3 u_3 & |a_3|-|a_4| & \rho_4 u_4 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$B(\boldsymbol{b}) := \begin{pmatrix} 1 & & & & & \\ & u_2 & & & & \\ & & \overline{u}_2 & & & \\ & & & u_4 & & \\ & & & & \overline{u}_4 & \\ & & & & & \ddots \end{pmatrix},$$

together with the fact that $A(\boldsymbol{a})$ is unitarily equivalent (by a diagonal transformation) to the Jacobi matrix

$$J(\boldsymbol{a}) := \begin{pmatrix} 1-|a_1| & \rho_1 & & & & \\ \rho_1 & |a_1|-|a_2| & \rho_2 & & & \\ & \rho_2 & |a_2|-|a_3| & \rho_3 & & \\ & & \rho_3 & |a_3|-|a_4| & \rho_4 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

show that $\|C(\boldsymbol{a}) - C(\boldsymbol{b})\| = \|J(\boldsymbol{a})\|$.

Since $J(\boldsymbol{a})$ is a bounded Jacobi matrix, it defines a self-adjoint operator on ℓ^2. So, it follows from Proposition 2.2.1 that $\sigma(J(\boldsymbol{a})) \subset \{z \in \mathbb{C} : \lim_n d(z, \sigma(J(a_1, \ldots, a_n))) = 0\}$, where $J(a_1, \ldots, a_n)$ is the principal submatrix of $J(\boldsymbol{a})$ of order n. A direct application of Gershgorin theorem shows that

$$\sigma(J(a_1, \ldots, a_n)) \subset \left\{ z \in \mathbb{C} : |z| \leqslant \max_{j=1}^{n} (\||a_{j-1}| - |a_j|\| + \rho_{j-1} + \rho_j) \right\},$$

where $a_0 = 1$ and $\rho_0 = 0$. Therefore,

$$\|C(\boldsymbol{a}) - C(\boldsymbol{b})\| = \|J(\boldsymbol{a})\| = \max_{\lambda \in \sigma(J(\boldsymbol{a}))} |\lambda| \leqslant \sup_{j \geqslant 0} (\||a_j| - |a_{j+1}|\| + \rho_j + \rho_{j+1}).$$

Taking into account that $|1-|a_{n+1}||+\rho_{n+1}\leqslant||a_n|-|a_{n+1}||+\rho_n+\rho_{n+1}$, a similar reasoning leads to

$$\|C(\boldsymbol{a}^{(n)})-C(\boldsymbol{b}^{(n)})\| = \|J(\boldsymbol{a}^{(n)})\|\leqslant \sup_{j\geqslant n}(||a_n|-|a_{n+1}||+\rho_n+\rho_{n+1}).$$

From this inequality we get

$$\inf_{n\geqslant 0}\|C(\boldsymbol{a}^{(n)})-C(\boldsymbol{b}^{(n)})\|\leqslant 2c_1(\boldsymbol{a}).$$

Summarizing, $\inf_{n\geqslant 0}\|C(\boldsymbol{a}^{(n)})-C(\boldsymbol{b}^{(n)})\|\leqslant 2c(\boldsymbol{a})$. Thus, Theorem 3.3 implies that a point $z\in\mathbb{T}$ is outside $\{\operatorname{supp}\mu\}'$ if $d(z,-\mathfrak{L}\{\overline{u}_n u_{n+1}\})>2c(\boldsymbol{a})$, which can be satisfied only if $c(\boldsymbol{a})<1$. In such a case we can write $c(\boldsymbol{a})=\cos\frac{\alpha}{2}$, $\alpha\in(0,\pi]$, and

$$\{\operatorname{supp}\mu\}'\subset\left\{z\in\mathbb{T}: d(z,-\mathfrak{L}\{\overline{u}_n u_{n+1}\})\leqslant 2\cos\tfrac{\alpha}{2}\right\}=\bigcup_{\lambda\in\mathfrak{L}\{\overline{u}_n u_{n+1}\}}\triangle_\alpha(\lambda). \qquad \square$$

An immediate corollary of this proposition is a condition for the Schur parameters which ensures that a certain arc of $\mathbb{T}$ is outside the derived set of the support of the measure.

Corollary 3.9. *Under the conditions of Proposition* 3.8,

$$\mathfrak{L}\{\overline{u}_n u_{n+1}\}\subset\overline{\Gamma}_\zeta(\lambda),\quad 0\leqslant\zeta<\alpha\quad\Rightarrow\quad\{\operatorname{supp}\mu\}'\subset\triangle_{\alpha-\zeta}(\lambda).$$

A remarkable consequence of Proposition 3.8 is obtained when studying measures μ in the class $\lim_n|a_{n+1}/a_n|=l$. Notice that $l\leqslant 1$ because $\boldsymbol{a}$ is bounded. When $l<1$, $\lim_n|a_n|=0$ and, thus, $\operatorname{supp}\mu=\mathbb{T}$ as a consequence of Weyl's theorem, since, for $b_n=0$, $C(\boldsymbol{a})-C(\boldsymbol{b})$ is compact. So, concerning $\operatorname{supp}\mu$, the only non-trivial case is $l=1$.

The condition $\lim_n|a_{n+1}/a_n|=1$ covers the case $\lim_n|a_n|=1$, for which $C(\boldsymbol{a})$ differs in a compact perturbation from a diagonal matrix with diagonal elements $-\overline{a}_{n-1}a_n$. Therefore, in this case, $\{\operatorname{supp}\mu\}'=-\mathfrak{L}\{a_{n+1}/a_n\}$ (see [13] for a similar argument using Hessenberg representations).

$\lim_n|a_{n+1}/a_n|=1$ is also verified when $\lim_n a_n=a\in\mathbb{D}\backslash\{0\}$, which implies that $C(\boldsymbol{a})-C(\boldsymbol{b})$ is compact for $b_n=a$. So, under this condition, $\{\operatorname{supp}\mu\}'=\triangle_\alpha(1)$, $\sin\frac{\alpha}{2}=|a|$, as in the Geronimus case corresponding to constant Schur parameters equal to a [10,15]. A bigger class is $\lim_n a_{n+1}/a_n=\lambda\in\mathbb{T}$, $\lim_n|a_n|=r\in(0,1)$, which is known as the López class [2]. It is a particular case of $\lim_n|a_{n+1}/a_n|=1$ too. In the López class, $C(\boldsymbol{a})$ is unitarily equivalent to a matrix obtained by a compact perturbation of the matrix $C(\boldsymbol{b})$ associated with the rotated Geronimus case $b_n=\lambda^n r$ (see [26, Chapter 4]) and, therefore, $\{\operatorname{supp}\mu\}'=\triangle_\alpha(\lambda)$, $\sin\frac{\alpha}{2}=r$.

All these results are known. We mention them to help understand to what extent the next theorem is an extension of them. Notice that, not only the López class, but also the two conditions that define this class are separately particular cases of $\lim_n|a_{n+1}/a_n|=1$.

Theorem 3.10. *If $\boldsymbol{a}$ is the sequence of Schur parameters associated with a measure μ on $\mathbb{T}$,*

$$\lim_n\left|\frac{a_{n+1}}{a_n}\right|=1\quad\Rightarrow\quad\{\operatorname{supp}\mu\}'\subset\bigcup_{\lambda\in\mathfrak{L}\left\{\frac{a_{n+1}}{a_n}\right\}}\triangle_\alpha(\lambda),\quad\sin\frac{\alpha}{2}=\underline{\lim}_n|a_n|.$$

Proof. The result is trivial when $\underline{\lim}_n |a_n| = 0$, so we just have to consider the case $\underline{\lim}_n |a_n| \neq 0$. The result follows from Proposition 3.8, taking into account that, if $\lim_n |a_{n+1}/a_n| = 1$ and $\lim_n |a_n| \neq 0$, $c(\boldsymbol{a}) = c_1(\boldsymbol{a}) = \overline{\lim}_n \rho_n < 1$ and $\mathfrak{L}\{\overline{u}_n u_{n+1}\} = \mathfrak{L}\{a_{n+1}/a_n\}$. □

Notice that the bounds provided by Theorem 3.10 give the exact location of $\{\operatorname{supp}\mu\}'$ for the López class and also for the case $\lim_n |a_n| = 1$. A direct consequence of this theorem is a result for the class defined only by the first of the López conditions.

Corollary 3.11. *If* $\boldsymbol{a}$ *is the sequence of Schur parameters associated with a measure* μ *on* $\mathbb{T}$,

$$\lim_n \frac{a_{n+1}}{a_n} = \lambda \in \mathbb{T} \quad \Rightarrow \quad \{\operatorname{supp}\mu\}' \subset \triangle_\alpha(\lambda), \quad \sin\frac{\alpha}{2} = \underline{\lim}_n |a_n|.$$

Theorem 3.10 can be used to supplement the conclusions of Theorem 3.4 and Corollary 3.5. Let us see an example.

Example 3.12. $\mathfrak{L}\{\boldsymbol{a}(\lambda)\} = \{a, b\}$, $\lambda \in \mathbb{T}$, $a \neq b$, $|a| = |b| \neq 0$.

When $\boldsymbol{a}(\lambda)$ has two different limit points a, b, using Corollary 3.5 we got information about an arc centred at λ which is free of $\{\operatorname{supp}\mu\}'$. Theorem 3.10 helps us find other arcs outside $\{\operatorname{supp}\mu\}'$ when $|a| = |b| \neq 0$ since, in this case, $\lim_n |\frac{a_{n+1}}{a_n}| = 1$.

Without loss of generality we can suppose $b = ae^{i\zeta}$, $\zeta \in (0, \pi]$, so, $\mathfrak{L}\{\frac{a_{n+1}}{a_n}\} \subset \{\lambda, \lambda e^{i\zeta}, \lambda e^{-i\zeta}\}$. Hence, Theorem 3.10 gives three possible arcs lying on $\mathbb{T}\backslash\{\operatorname{supp}\mu\}'$, centred at λ and $-\lambda e^{\pm i\frac{\zeta}{2}}$. More precisely,

$$\sin\frac{\zeta}{2} < |a| \;\Rightarrow\; \{\operatorname{supp}\mu\}' \subset \triangle_{\alpha_1}(\lambda),\; \alpha_1 = \alpha - \zeta,$$
$$\cos\frac{\zeta}{4} < |a| \;\Rightarrow\; \{\operatorname{supp}\mu\}' \subset \triangle_{\alpha_2}(-\lambda e^{i\frac{\zeta}{2}}) \cap \triangle_{\alpha_2}(-\lambda e^{-i\frac{\zeta}{2}}),\; \alpha_2 = \alpha + \frac{\zeta}{2} - \pi,$$

where $\alpha \in (0, \pi)$ is given by $\sin\frac{\alpha}{2} = |a|$.

Concerning Proposition 3.8, when $\lim_n(|a_{n+1}| - |a_n|) = 0$, $c(\boldsymbol{a}) = c_1(\boldsymbol{a})$ since $\rho_n < \sqrt{2(1-|a_n|)}$. On the contrary, $c(\boldsymbol{a}) = c_2(\boldsymbol{a})$ if $\lim_n |a_{2n-1}| = 1$ or $\lim_n |a_{2n}| = 1$, due to the inequality $1 - |a_n| + \rho_n > \sqrt{2(1-|a_n|)}$. So, in the last case, $c(\boldsymbol{a}) = \overline{\lim}_n \sqrt{(1-|a_n|)/2}$ and we get the following corollary, which is a generalization of a result given in [13, p. 72].

Corollary 3.13. *Let* $a_n = |a_n|u_n$ $(u_n \in \mathbb{T})$ *be the Schur parameters of a measure* μ *on* $\mathbb{T}$. *If* $\lim_n |a_{2n-1}| = 1$ *or* $\lim_n |a_{2n}| = 1$, *then,*

$$\{\operatorname{supp}\mu\}' \subset \bigcup_{\lambda \in \mathfrak{L}\{\overline{u}_n u_{n+1}\}} \triangle_\alpha(\lambda), \quad \cos\alpha = -\underline{\lim}_n |a_n|.$$

The rotated Geronimus case $C(\boldsymbol{b})$, $b_n = \lambda^n a$, $a \in \mathbb{D}\backslash\{0\}$, $\lambda \in \mathbb{T}$, can also be used for the comparison in Theorem 3.3, since we know that $\sigma_e(C(\boldsymbol{b})) = \triangle_\alpha(\lambda)$, $\sin\frac{\alpha}{2} = |a|$. In fact, the consequences of this comparison are a particularization of a more general result concerning the comparison with the case $b_{2n-1} = \lambda^{2n-1}a_o$, $b_{2n} = \lambda^{2n}a_e$, where $a_o, a_e \in \overline{\mathbb{D}}$. In this case it is known that $\sigma_e(C(\boldsymbol{b})) = \triangle_{\alpha_+}(\lambda) \cap \triangle_{\alpha_-}(-\lambda)$ where $\alpha_\pm \in [0, \pi]$ are given by

$$\cos\alpha_\pm = \rho_o\rho_e \mp \operatorname{Re}(\overline{a}_o a_e), \quad \rho_i := \sqrt{1-|a_i|^2}, \quad i = o, e \tag{18}$$

(see [11,24]). That is, $\sigma_e(C(\boldsymbol{b}))$ has two connected components except for the cases $a_o = \pm a_e$ which correspond to only one connected component.

Proposition 3.14. *Let* $\boldsymbol{a}$ *be the sequence of Schur parameters of a measure* μ *on* $\mathbb{T}$ *and, for* $a_o, a_e \in \overline{\mathbb{D}}$ *and* $\lambda \in \mathbb{T}$, *let us define*

$$s(\boldsymbol{a}) := \frac{1}{2}\left\{\overline{\lim_{\text{odd}\, n}}\; k(a_n(\lambda), a_o) + \overline{\lim_{\text{even}\, n}}\; k(a_n(\lambda), a_e)\right\},$$

with $k(\cdot,\cdot)$ *given in* (16). Then, defining $\alpha_\pm$ as in (18),

$$s(\boldsymbol{a}) < \sin\frac{\alpha_\pm}{2} \quad\Rightarrow\quad \{\operatorname{supp}\mu\}' \subset \triangle_{\alpha_\pm-\zeta}(\pm\lambda), \qquad \sin\frac{\zeta}{2} = s(\boldsymbol{a}), \qquad 0 \leqslant \zeta < \alpha_\pm.$$

Proof. Let us consider the sequence $b_{2n-1} := \lambda^{2n-1}a_o$, $b_{2n} := \lambda^{2n}a_e$. Using (15) we find that

$$\inf_{n\geqslant 0} \|C(\boldsymbol{a}^{(n)}) - C(\boldsymbol{b}^{(n)})\| \leqslant 2s(\boldsymbol{a}).$$

When $s(\boldsymbol{a}) < \sin\frac{\alpha_\pm}{2}$ we can write $s(\boldsymbol{a}) = \sin\frac{\zeta}{2}$, $\zeta \in [0, \alpha_\pm)$, and, since $\sigma_e(C(\boldsymbol{b})) = \triangle_{\alpha_+}(\lambda) \cap \triangle_{\alpha_-}(-\lambda)$, Theorem 3.3 implies that

$$\{\operatorname{supp}\mu\}' \subset \left\{z \in \mathbb{T} : d(z, \triangle_{\alpha_\pm}(\pm\lambda)) \leqslant 2\sin\frac{\zeta}{2}\right\} = \triangle_{\alpha_\pm-\zeta}(\pm\lambda). \qquad \square$$

Remark 3.15. With the notation of (16), from $y_1^2 - y_2^2 = |x_2|^2 - |x_1|^2$, we get

$$|y_1 - y_2| \leqslant \frac{|x_1| + |x_2|}{y_1 + y_2}\,|x_1 - x_2|,$$

hence,

$$k(x_1, x_2) \leqslant \frac{\sqrt{2(1 + |x_1|\,|x_2| + y_1y_2)}}{y_1 + y_2}\,|x_1 - x_2| \leqslant \frac{2}{y_1 + y_2}\,|x_1 - x_2|.$$

Therefore,

$$k(a_n(\lambda), a_i) < \frac{2}{\rho_i}\,|a_n(\lambda) - a_i|,$$

and the conclusions of Proposition 3.14 hold if

$$\frac{1}{\rho_o}\overline{\lim_{\text{odd}\, n}}\; |a_n(\lambda) - a_o| + \frac{1}{\rho_e}\overline{\lim_{\text{even}\, n}}\; |a_n(\lambda) - a_e| < \sin\frac{\alpha_\pm}{2}.$$

Another subclass of $\lim_n |a_{n+1}/a_n| = 1$ is given by the second López condition, $\lim_n |a_n| = r \in (0, 1)$. In this subclass, Proposition 3.14 supplements Theorem 3.10 with the result that we present below. Notice that $\lim_n |a_n| = r$ if and only if, for $\lambda \in \mathbb{T}$, the limit points of $\boldsymbol{a}(\lambda)$ lie on the circle $\{z \in \mathbb{C} : |z| = r\}$. The key idea is that the knowledge of the arcs of this circle in which the limit points of $\boldsymbol{a}(\lambda)$ lie, gives information about the arc of $\mathbb{T}$ around λ that is free of $\{\operatorname{supp}\mu\}'$. We will state a more general result that deals with the case $\lim_n |a_{2n-1}(\lambda)| = r_o$, $\lim_n |a_{2n}(\lambda)| = r_e$.

Theorem 3.16. *Let* $\boldsymbol{a}$ *be the sequence of Schur parameters of a measure* μ *on* $\mathbb{T}$. *Assume that for some* $\lambda \in \mathbb{T}$ *the limit points of the odd and even subsequences of* $\boldsymbol{a}(\lambda)$ *lie on* $\overline{\Gamma}_{\xi_o}(a_o)$ *and* $\overline{\Gamma}_{\xi_e}(a_e)$, *respectively. Then, if* $\alpha_\pm$ *is given in* (18),

$$s := |a_o| \sin \frac{\xi_o}{2} + |a_e| \sin \frac{\xi_e}{2} < \sin \frac{\alpha_\pm}{2} \quad \Rightarrow \quad \{\operatorname{supp} \mu\}' \subset \Delta_{\alpha_\pm - \zeta}(\pm\lambda), \quad \sin \frac{\zeta}{2} = s.$$

Proof. Since $\lim_n |a_{2n-1}| = |a_o|$ and $\lim_n |a_{2n}| = |a_e|$,

$$s(\boldsymbol{a}) = \frac{1}{2} \left\{ \overline{\lim_{\text{odd}\, n}} \; |a_n(\lambda) - a_o| + \overline{\lim_{\text{even}\, n}} \; |a_n(\lambda) - a_e| \right\}.$$

The statement follows from Proposition 3.14 and the fact that, when the limit points of $(a_n(\lambda))_{n \geqslant 1}$ lie on $\overline{\Gamma}_\xi(a)$,

$$\overline{\lim_n} \, |a_n(\lambda) - a| \leqslant 2|a| \sin \frac{\xi}{2}. \qquad \square$$

Remark 3.17. The class $\lim_n |a_n| = r \in (0, 1)$ corresponds to the case $|a_o| = |a_e| \neq 0, 1$. Then, $\{a_o, a_e\} = \{a, ae^{i\omega}\}$ with $|a| = r$ and $\omega \in [0, \pi]$, so, $\sin \frac{\alpha_+}{2} = r \cos \frac{\omega}{2}$ and $\sin \frac{\alpha_-}{2} = r \sin \frac{\omega}{2}$. Hence, in this case, the previous Theorem gives

$$\eta := \sin \frac{\xi_o}{2} + \sin \frac{\xi_e}{2} < \begin{cases} \cos \frac{\omega}{2} \\ \sin \frac{\omega}{2} \end{cases} \quad \Rightarrow \quad \{\operatorname{supp} \mu\}' \subset \Delta_{\alpha_\pm - \zeta}(\pm\lambda), \quad \sin \frac{\zeta}{2} = \eta r.$$

4. Finite truncations of $C(\boldsymbol{a})$ and the support of the measure

Given an operator on an infinite-dimensional Hilbert space, the search for finite truncations whose spectra asymptotically approach the spectrum of the full operator is an old and non-trivial problem. For our purposes, the relevant question is, if, given a sequence $\boldsymbol{a}$ in $\mathbb{D}$, there exist sequences of finite truncations of the unitary matrix $C(\boldsymbol{a})$ such that their spectra approximate to the spectrum of $C(\boldsymbol{a})$, that is, to the support of the related measure on $\mathbb{T}$ (for the analogous problem concerning Jacobi matrices and measures on the real line see [3,5,19]). We will see that the normal truncations of $C(\boldsymbol{a})$ on ℓ_n^2 give a positive answer to this question.

At this point we have to remember the definitions of $\overline{\lim}_n$, $\underline{\lim}_n$ and $\lim_n$ for sequences of subsets of $\mathbb{C}$, in the sense of Hahn [17] or Kuratowski [22].

Definition 4.1. Given a sequence $\boldsymbol{E} = (E_n)_{n \geqslant 1}$, $E_n \subset \mathbb{C}$,

$$\overline{\lim_n} \; E_n := \left\{ \lambda \in \mathbb{C} : \underline{\lim_n} \, d(\lambda, E_n) = 0 \right\},$$

$$\underline{\lim_n} \; E_n := \left\{ \lambda \in \mathbb{C} : \lim_n d(\lambda, E_n) = 0 \right\},$$

$$E = \lim_n E_n \text{ iff } \underline{\lim_n} \, E_n = E = \overline{\lim_n} \, E_n.$$

The points in $\overline{\lim}_n E_n$ are called the (weak) limit points of $\boldsymbol{E}$, while the points in $\underline{\lim}_n E_n$ are called the strong limit points of $\boldsymbol{E}$.

$\underline{\lim}_n E_n$ and $\overline{\lim}_n E_n$ are closed sets such that $\underline{\lim}_n E_n \subset \overline{\lim}_n E_n$. The points in $\underline{\lim}_n E_n$ are the limits of the convergent sequences $(\lambda_n)_{n \geqslant 1}$ with $\lambda_n \in E_n$, $\forall n \geqslant 1$, while $\overline{\lim}_n E_n$ contains the strong limit points of all the subsequences of $\boldsymbol{E}$. We have the following relations.

Lemma 4.2. *For any* $z \in \mathbb{C}$,

$$d\left(z, \overline{\lim_n} E_n\right) = \underline{\lim_n}\, d(z, E_n) \leqslant \overline{\lim_n}\, d(z, E_n) \leqslant d\left(z, \underline{\lim_n} E_n\right).$$

Proof. If $\lambda \in \underline{\lim}_n E_n$, there exists $(\lambda_n)_{n\geqslant 1}$, $\lambda_n \in E_n$, $\forall n \geqslant 1$, such that $\lambda = \lim_n \lambda_n$. Therefore, $d(z,\lambda) = \lim_n d(z,\lambda_n) \geqslant \overline{\lim}_n d(z, E_n)$. Hence, $d(z, \underline{\lim}_n E_n) \geqslant \overline{\lim}_n d(z, E_n)$.

A similar argument shows that $d(z, \overline{\lim}_n E_n) \geqslant \underline{\lim}_n d(z, E_n)$. So, in the case $\underline{\lim}_n d(z, E_n) = \infty$ the relation is proved. Otherwise, let $(E_n)_{n\in\mathcal{I}}$ be a subsequence of $\boldsymbol{E}$ such that $\underline{\lim}_n d(z, E_n) = \lim_{n\in\mathcal{I}} d(z, E_n)$. Then, $\underline{\lim}_n d(z, E_n) = \lim_{n\in\mathcal{I}} d(z, \lambda_n)$, $\lambda_n \in E_n$, $\forall n \in \mathcal{I}$. Since $(\lambda_n)_{n\in\mathcal{I}}$ must be bounded, it has a convergent subsequence $(\lambda_n)_{n\in\mathcal{J}}$. Therefore, $\lambda = \lim_{n\in\mathcal{J}} \lambda_n \in \overline{\lim}_n E_n$ and $d(z, \overline{\lim}_n E_n) \leqslant d(z,\lambda) = \underline{\lim}_n d(z, E_n)$. □

Given a sequence $(T_n)_{n\geqslant 1}$ of truncations of an operator T, we are interested in the limit and strong limit points of the related spectra, that is, $\overline{\lim}_n \sigma(T_n)$ and $\underline{\lim}_n \sigma(T_n)$. With this notation, Proposition 2.1.1 says that, if T is a bounded normal band operator on ℓ^2 and $(T_n)_{n\geqslant 1}$ is a bounded sequence of normal truncations of T, T_n being a truncation on ℓ^2_n, then $\sigma(T) \subset \underline{\lim}_n \sigma(T_n)$.

Concerning the relation between the limit points of the spectra for different sequences of truncations, we have the following result.

Proposition 4.3. *For* $n \geqslant 1$, *let* T_n, T'_n *be bounded truncations of a given operator on the same subspace. If the truncations* T_n *are normal, then*

$$\overline{\lim_n}\, \sigma(T'_n) \subset \left\{ z \in \mathbb{C} : d\left(z, \overline{\lim_n}\, \sigma(T_n)\right) \leqslant \overline{\lim_n}\, \|T'_n - T_n\| \right\}.$$

Proof. Let $\lambda \in \overline{\lim}_n \sigma(T'_n)$. There exists $(\lambda_n)_{n\in\mathcal{I}}$, $\mathcal{I} \subset \mathbb{N}$, with $\lambda_n \in \sigma(T'_n)$, $\forall n \in \mathcal{I}$, and $\lambda = \lim_{n\in\mathcal{I}} \lambda_n$. Since T_n is normal, $d(\lambda_n, \sigma(T_n)) \leqslant \|T'_n - T_n\|$ and, so, $d(\lambda, \sigma(T_n)) \leqslant |\lambda - \lambda_n| + \|T'_n - T_n\|$. Therefore, using Lemma 4.2 we get $d(\lambda, \overline{\lim}_n \sigma(T_n)) = \underline{\lim}_n d(\lambda, \sigma(T_n)) \leqslant \overline{\lim}_n \|T'_n - T_n\|$. □

Given a sequence $\boldsymbol{a} = (a_n)_{n\geqslant 1}$ in $\mathbb{D}$ and a sequence $\boldsymbol{u} = (u_n)_{n\geqslant 1}$ in $\mathbb{T}$, we can consider the corresponding sequence $(C(a_1, \ldots, a_{n-1}, u_n))_{n\geqslant 1}$ of finite unitary truncations of $C(\boldsymbol{a})$. Our aim is to study the relation between the limit and strong limit points of the spectra of these truncations and the spectrum of $C(\boldsymbol{a})$. The spectrum of $C(a_1, \ldots, a_{n-1}, u_n)$ is the set of zeros of the para-orthogonal polynomial $p_n^{u_n}$ associated with the measure related to $\boldsymbol{a}$. This means that, in fact, we are going to study the connection between the support of a measure on $\mathbb{T}$ and the limit and strong limit points of the zeros of sequences $(p_n^{u_n})_{n\geqslant 1}$ of para-orthogonal polynomials associated with this measure. Some previous results in this direction can be found in [6,14]. For convenience, in what follows we use the notation

$$\Sigma_n(\boldsymbol{a}; \boldsymbol{u}) := \sigma(C(a_1, \ldots, a_{n-1}, u_n)) = \{z \in \mathbb{C} : p_n^{u_n}(z) = 0\},$$

for any sequence $\boldsymbol{a}$ in $\mathbb{D}$ and $\boldsymbol{u}$ in $\mathbb{T}$. The results achieved till now have the following consequences.

Theorem 4.4. *If* $\boldsymbol{a}$ *is the sequence of Schur parameters associated with a measure* μ *on* $\mathbb{T}$ *and* $\boldsymbol{u}$ *is an arbitrary sequence in* $\mathbb{T}$,

$$\operatorname{supp} \mu \subset \underline{\lim_n}\, \Sigma_n(\boldsymbol{a}; \boldsymbol{u}).$$

Moreover, for any other sequence $\boldsymbol{u}'$ *in* $\mathbb{T}$,

$$\overline{\lim_n}\, \Sigma_n(\boldsymbol{a}; \boldsymbol{u}') \subset \left\{ z \in \mathbb{T} : d\left(z, \overline{\lim_n}\, \Sigma_n(\boldsymbol{a}; \boldsymbol{u})\right) \leqslant \overline{\lim_n}\, |u_n' - u_n| \right\},$$

and, so,

$$\lim_n (u_n' - u_n) = 0 \quad \Rightarrow \quad \overline{\lim_n}\, \Sigma_n(\boldsymbol{a}; \boldsymbol{u}') = \overline{\lim_n}\, \Sigma_n(\boldsymbol{a}; \boldsymbol{u}).$$

Proof. The first statement is a direct consequence of Proposition 2.1.1.

The second one follows from Proposition 4.3, taking into account that $\|C(a_1, \ldots, a_{n-1}, u_n') - C(a_1, \ldots, a_{n-1}, u_n)\| = |u_n' - u_n|$, as can be easily proved using $C(a_1, \ldots, a_{n-1}, u_n) = C_{\mathrm{o}}(a_1, \ldots, a_{n-1}, u_n) C_{\mathrm{e}}(a_1, \ldots, a_{n-1}, u_n)$. □

The fact that the strong limit points of the zeros of para-orthogonal polynomials include the support of the orthogonality measure was already proved in [14, Theorem 8], which deals with sequences of para-orthogonal polynomials with a fixed zero. We have obtained the result as a particular case of a more general statement of operator theory. This result will be improved later (see Theorems 4.17, 4.18 and Corollary 4.19) although we cannot always expect a strict equality between those strong limit points and the support of the measure, due to the freedom in one of the zeros for the para-orthogonal polynomials of a given order (see Remark 2.3).

Concerning the weak limit points of the zeros of para-orthogonal polynomials, it was also shown in [14, Examples 9 and 10] that some of them can lie outside the support of the measure, even if we fix for all the para-orthogonal polynomials a common zero inside the support of the measure. However, we can get some information about the location of these limit points, which will be useful for the study of the convergence of rational approximants for the Carathéodory function of the measure (see Section 5). The next theorem is an example of this kind of results. If there is a limit point outside the support of the measure, this theorem establishes how far it can be from the derived set of this support. The proof, which follows the ideas of [19, Theorem 2.3] relating to orthogonal truncations of self-adjoint operators, needs the following lemmas.

Lemma 4.5. *Let* $T_0, T \in \mathfrak{B}(H)$ *be normal and such that* $T - T_0$ *is compact. If* $T_n := T[Q_n]$ *is a finite normal truncation of T for* $n \geqslant 1$ *and* $\hat{T}_n \to T$,

$$\sup_{\lambda \in \overline{\lim}_n \sigma(T_n) \setminus \sigma(T)} |\lambda - z| \leqslant \overline{\lim_n}\, \|Q_n\| \sup_{\lambda \in \sigma(T_0)} |\lambda - z|, \quad \forall z \in \mathbb{C}.$$

Proof. Let $\lambda \in \overline{\lim}_n \sigma(T_n) \setminus \sigma(T)$. Then, there exists $(\lambda_n)_{n \in \mathcal{I}}$, $\mathcal{I} \subset \mathbb{N}$, such that $\lim_{n \in \mathcal{I}} \lambda_n = \lambda$ and λ_n is an eigenvalue of the finite truncation T_n. Let x_n be a unitary eigenvector of T_n with eigenvalue λ_n. Since T_n is normal, we can suppose that x_n is also an eigenvector of T_n^* with eigenvalue $\overline{\lambda}_n$. So, given an arbitrary $y \in H$,

$$(x_n, (\lambda - T)y) = ((\overline{\lambda} - \hat{T}_n^*)x_n, y) + (x_n, (\hat{T}_n - T)y),$$

which gives

$$|(x_n, (\lambda - T)y)| \leqslant |\lambda - \lambda_n| \|y\| + \|(\hat{T}_n - T)y\|.$$

$(x_n)_{n \in \mathcal{I}}$ is bounded, thus, there exists a subsequence $(x_n)_{n \in \mathcal{J}}$ weakly converging to some $x \in H$. Taking limits in the above inequality for $n \in \mathcal{J}$ we get $((\overline{\lambda} - T^*)x, y) = 0$, $\forall y \in H$, that is, $(\overline{\lambda} - T^*)x = 0$. Since T is normal and $\lambda \notin \sigma(T)$, $\overline{\lambda}$ is not an eigenvalue of T^*, thus, $x = 0$.

Let $T_{0n} := T_0[Q_n]$. Then, for any $z \in \mathbb{C}$, we can write

$$(\lambda - z)x_n = (T_{0n} - z)x_n + (T_n - T_{0n})x_n + (\lambda - \lambda_n)x_n,$$

and, hence,

$$|\lambda - z| \leqslant \|Q_n\|\|T_0 - z\| + \|Q_n\|\|(T - T_0)x_n\| + |\lambda - \lambda_n|.$$

The fact that $(x_n)_{n\in\mathcal{J}}$ weakly converges to 0 and $T - T_0$ is compact implies that $\lim_{n\in\mathcal{J}} \|(T - T_0)x_n\| = 0$. We can suppose $\overline{\lim}_n \|Q_n\| < \infty$, otherwise the inequality of the theorem is trivial. Then, taking limits in the last inequality for $n \in \mathcal{J}$ we obtain

$$|\lambda - z| \leqslant \overline{\lim_n} \|Q_n\|\|T_0 - z\|.$$

T_0 is normal, hence, $\|T_0 - z\| = \sup_{\lambda\in\sigma(T_0)} |\lambda - z|$. So, the theorem is proved since λ was an arbitrary point in $\overline{\lim}_n \sigma(T_n)\backslash\sigma(T)$. $\square$

The previous result for normal operators has the following consequence in the special case of unitary operators.

Lemma 4.6. *Let U be a unitary operator on H and $U_n := U[Q_n]$ be a finite unitary truncation of U for $n \geqslant 1$ such that $\hat{U}_n \to U$. Define $\alpha_0 \in [0, \pi]$ by*

$$\cos\frac{\alpha_0}{2} = \underline{\lim_n} \frac{1}{\|Q_n\|}.$$

Then, if $\sigma_e(U) \subset \Delta_\alpha(w)$,

$$\alpha > \alpha_0 \quad \Rightarrow \quad \overline{\lim_n}\, \sigma(U_n)\backslash\sigma(U) \subset \Delta_\beta(w), \quad \cos\frac{\beta}{2} = \frac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha_0}{2}}.$$

Proof. Let us suppose that $\sigma_e(U) \subset \Delta_\alpha(w)$. Then, for any $\varepsilon \in (0, \alpha)$, $S_\varepsilon := E_U(\Gamma_\varepsilon(w))H$ has finite dimension. Thus, the unitary operator

$$U^\varepsilon := U_{\restriction S_\varepsilon^\perp} \oplus (-w1_{\restriction S_\varepsilon})$$

differs from U in a finite rank perturbation and, so, $U - U^\varepsilon$ is compact. Moreover, $\sigma(U^\varepsilon) \subset \Delta_\varepsilon(w)$. Hence, Lemma 4.5 gives

$$\sup_{\lambda\in\overline{\lim}_n \sigma(U_n)\backslash\sigma(U)} |\lambda + w| \leqslant \overline{\lim_n} \|Q_n\| \sup_{\lambda\in\sigma(U^\varepsilon)} |\lambda + w| \leqslant 2\frac{\cos\frac{\varepsilon}{2}}{\cos\frac{\alpha_0}{2}}, \quad \forall\varepsilon \in (0, \alpha).$$

If $\alpha > \alpha_0$, then $\dfrac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha_0}{2}} = \cos\dfrac{\beta}{2}$, $\beta \in (0, \pi]$. Thus, from the above inequality,

$$\sup_{\lambda\in\overline{\lim}_n \sigma(U_n)\backslash\sigma(U)} |\lambda + w| \leqslant 2\cos\frac{\beta}{2},$$

which proves the result. $\square$

Now we can get the announced result about the limit points of the zeros of para-orthogonal polynomials.

Theorem 4.7. *Let $\boldsymbol{a}$ be the sequence of Schur parameters of a measure μ on $\mathbb{T}$, $\{\Gamma_{\alpha_j}(w_j)\}_{j=1}^N$ ($N \in \mathbb{N} \cup \{\infty\}$) being the connected components of $\mathbb{T}\backslash\{\operatorname{supp}\mu\}'$. Let $\boldsymbol{u}$ be a sequence in $\mathbb{T}$ and define $\alpha_0 \in [0, \pi]$ by*

$$\cos\frac{\alpha_0}{2} = \varliminf_n \frac{1}{\sqrt{1 + \frac{|u_n - a_n|^2}{\rho_n^2}}}.$$

Then, for any $j = 1, \ldots, N$,

$$\alpha_j > \alpha_0 \quad \Rightarrow \quad \varlimsup_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) \cap \Gamma_{\beta_j}(w_j) = \operatorname{supp}\mu \cap \Gamma_{\beta_j}(w_j), \quad \cos\frac{\beta_j}{2} = \frac{\cos\frac{\alpha_j}{2}}{\cos\frac{\alpha_0}{2}}.$$

Proof. Since $\operatorname{supp}\mu \subset \varliminf_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) \subset \varlimsup_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u})$, we just have to prove that $\varlimsup_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u})\backslash \operatorname{supp}\mu \subset \triangle_{\beta_j}(w_j)$ for each j such that $\alpha_j > \alpha_0$. Remember that $C(a_1, \ldots, a_{n-1}, u)$ is the unitary truncation of $C(\boldsymbol{a})$ on ℓ_n^2 associated with the projection $Q_n(\boldsymbol{a}; u)$ whose norm is given in Corollary 2.4. From Proposition 2.1.1, $\hat{C}(a_1, \ldots, a_{n-1}, u) \to C(\boldsymbol{a})$. So, if $\alpha_j > \alpha_0$, a direct application of Lemma 4.6 with $U = C(\boldsymbol{a})$ and $Q_n = Q_n(\boldsymbol{a}; u_n)$ gives $\varlimsup_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u})\backslash\operatorname{supp}\mu \subset \triangle_{\beta_j}(w_j)$. □

The previous theorem says that $\varlimsup_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u})$ can differ from $\operatorname{supp}\mu$ only in $\Gamma_{\alpha_j}(w_j)\backslash\Gamma_{\beta_j}(w_j)$ if $\alpha_j > \alpha_0$, or in $\Gamma_{\alpha_j}(w_j)$ if $\alpha_j \leqslant \alpha_0$.

If $a_n \neq 0$ for any n big enough, the best choice for the sequence $\boldsymbol{u}$ in Theorem 4.7 is $u_n = \frac{a_n}{|a_n|}$. Then,

$$\cos\alpha_0 = \varliminf_n |a_n|,$$

so, $\alpha_0 \leqslant \frac{\pi}{2}$. Taking into account Theorem 4.4, Theorem 4.7 also works with the above value of α_0 if $\lim_n(u_n - \frac{a_n}{|a_n|}) = 0$. In particular, if $\lim_n |a_n| = 1$ we can choose $\boldsymbol{u}$ such that $\lim_n(u_n - a_n) = 0$, which gives $\alpha_0 = 0$ and, hence, $\operatorname{supp}\mu = \varliminf_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) = \varlimsup_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u})$. So we get the following consequence of Theorem 4.7.

Corollary 4.8. *If $\boldsymbol{a}$ is the sequence of Schur parameters associated with a measure μ on $\mathbb{T}$ and $\boldsymbol{u}$ is a sequence in $\mathbb{T}$,*

$$\lim_n(u_n - a_n) = 0 \quad \Rightarrow \quad \lim_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) = \operatorname{supp}\mu.$$

As we pointed out, $\{\operatorname{supp}\mu\}' = -\mathfrak{L}\{a_{n+1}/a_n\}$ when $\lim_n |a_n| = 1$, so, if $\lim_n(u_n - a_n) = 0$, $\lim_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u})$ coincides with $-\mathfrak{L}\{a_{n+1}/a_n\}$ plus, at most, a countable set that can accumulate only on $-\mathfrak{L}\{a_{n+1}/a_n\}$.

Example 4.9. Rotated asymptotically 2-periodic Schur parameters.

Let us suppose that $\lim_n a_{2n-1}(\lambda) = a_{\mathrm{o}}$ and $\lim_n a_{2n}(\lambda) = a_{\mathrm{e}}$ for some $\lambda \in \mathbb{T}$. We know that $\mathbb{T}\backslash\{\operatorname{supp}\mu\}' = \Gamma_{\alpha_+}(\lambda) \cup \Gamma_{\alpha_-}(-\lambda)$ where $\alpha_\pm$ is given in (18). If $u_n = \frac{a_n}{|a_n|}$, then $\cos\alpha_0 =$

$\min\{|a_\mathrm{o}|, |a_\mathrm{e}|\}$ and

$$\alpha_\pm > \alpha_0 \Leftrightarrow \rho_\mathrm{o}\rho_\mathrm{e} \mp \mathrm{Re}(\overline{a}_\mathrm{o} a_\mathrm{e}) < \min\{|a_\mathrm{o}|, |a_\mathrm{e}|\}.$$

Under these conditions, $\overline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) \cap \Gamma_{\beta_\pm}(\pm\lambda) = \operatorname{supp}\mu \cap \Gamma_{\beta_\pm}(\pm\lambda)$ where

$$\cos\frac{\beta_\pm}{2} = \sqrt{\frac{1 + \rho_\mathrm{o}\rho_\mathrm{e} \mp \mathrm{Re}(\overline{a}_\mathrm{o} a_\mathrm{e})}{1 + \min\{|a_\mathrm{o}|, |a_\mathrm{e}|\}}}.$$

Example 4.10. $\mathfrak{L}\{\boldsymbol{a}(\lambda)\} = \{a, b\}$, $\lambda \in \mathbb{T}$, $a \neq b$, $|a| = |b| \neq 0$.

Following Example 3.12, we can suppose $b = ae^{i\zeta}$, $\zeta \in (0, \pi]$, and, then

$$\Gamma_{\alpha_1}(\lambda) \subset \mathbb{T}\backslash\{\operatorname{supp}\mu\}', \quad \alpha_1 = \alpha - \zeta, \quad \text{if } \sin\frac{\zeta}{2} < |a|,$$
$$\Gamma_{\alpha_2}(-\lambda e^{\pm i\frac{\zeta}{2}}) \subset \mathbb{T}\backslash\{\operatorname{supp}\mu\}', \quad \alpha_2 = \alpha + \frac{\zeta}{2} - \pi, \quad \text{if } \cos\frac{\zeta}{4} < |a|,$$

where $\alpha \in (0, \pi)$ is given by $\sin\frac{\alpha}{2} = |a|$. Let $u_n = \frac{a_n}{|a_n|}$, so that $\cos\alpha_0 = |a|$, that is, $\alpha_0 = \alpha - \zeta_0$ with

$$\sin\frac{\zeta_0}{2} = (2|a| - 1)\sqrt{\frac{1 + |a|}{2}}, \quad -\frac{\pi}{2} < \zeta_0 < \pi.$$

Hence, we get

$$\alpha_1 > \alpha_0 \Leftrightarrow \zeta < \zeta_0 \Leftrightarrow \sin\frac{\zeta}{2} < (2|a| - 1)\sqrt{\frac{1 + |a|}{2}},$$
$$\alpha_2 > \alpha_0 \Leftrightarrow \pi - \frac{\zeta}{2} < \zeta_0 \Leftrightarrow \cos\frac{\zeta}{4} < (2|a| - 1)\sqrt{\frac{1 + |a|}{2}}.$$

Under each of these conditions, $\overline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) \cap \Gamma_{\beta_1}(\lambda) = \operatorname{supp}\mu \cap \Gamma_{\beta_1}(\lambda)$ and $\overline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) \cap \Gamma_{\beta_2}(-\lambda e^{\pm i\frac{\zeta}{2}}) = \operatorname{supp}\mu \cap \Gamma_{\beta_2}(-\lambda e^{\pm i\frac{\zeta}{2}})$, respectively, where

$$\cos\frac{\beta_1}{2} = \frac{|a|\,\sin\frac{\zeta}{2} + \rho\cos\frac{\zeta}{2}}{\sqrt{\frac{1+|a|}{2}}}, \quad \cos\frac{\beta_2}{2} = \frac{|a|\cos\frac{\zeta}{4} + \rho\sin\frac{\zeta}{4}}{\sqrt{\frac{1+|a|}{2}}},$$

being $\rho = \sqrt{1 - |a|^2}$.

We can go further in the analysis of $\overline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u})$ and $\underline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u})$ using the analytic properties of the para-orthogonal polynomials. The following result for the corresponding zeros was proved in [6,14].

Theorem A (*Cantero et al. [6, Corollary 2] and Golinskii [14, Theorem 2]*). *Given a measure μ on $\mathbb{T}$, the closure $\overline{\Gamma}$ of any arc $\Gamma \subset \mathbb{T}\backslash\operatorname{supp}\mu$ contains at most one zero of the para-orthogonal polynomial p_n^u related to μ for any $u \in \mathbb{T}$ and $n \in \mathbb{N}$.*

With this property and Theorem 4.4 we can achieve the following result.

Theorem 4.11. *Let μ be a measure on $\mathbb{T}$ with a sequence $\boldsymbol{a}$ of Schur parameters, and let $\boldsymbol{u}$ be a sequence in $\mathbb{T}$. Consider a connected component Γ of $\mathbb{T}\backslash\{\operatorname{supp}\mu\}'$ and $w \in \Gamma$.*

1. $w \in \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) \forall n \geqslant 1 \Rightarrow \overline{\lim_n} \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) \cap \Gamma = (\operatorname{supp}\mu \cap \Gamma) \cup \{w\}$.
2. $w \in \underline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) \backslash \operatorname{supp}\mu \Rightarrow \overline{\lim_n} \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) \cap \Gamma = (\operatorname{supp}\mu \cap \Gamma) \cup \{w\}$.

Proof. $\Gamma \cap \operatorname{supp}\mu$ is at most a countable set which can accumulate only at the endpoints of $\overline{\Gamma}$. Consider one of the two connected components of $\Gamma\backslash\{w\}$, let us say $\Gamma_+ = (w, w_+)$. Let $\Gamma_+\backslash\operatorname{supp}\mu = (w, w_1) \cup (w_1, w_2) \cup \cdots$ be the decomposition in connected components. From Theorem A, it is clear that (w_j, w_{j+1}) has, at most, one point in $\Sigma_n(\boldsymbol{a}; \boldsymbol{u})$ for each $j, n \geqslant 1$.

Assume that $w \in \Sigma_n(\boldsymbol{a}; \boldsymbol{u})$, $\forall n \geqslant 1$. Then, Theorem A implies that $(w, w_1) \cap \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) = \emptyset$, $\forall n \geqslant 1$. Hence, for $j \geqslant 1$, since $w_j \in \operatorname{supp}\mu \subset \underline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u})$, it is necessary that $(w_j, w_{j+1}) \cap \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) = \{\lambda_j^{(n)}\}$ for n greater than certain index n_j, and $\lim_n \lambda_j^{(n)} = w_j$. So, we conclude that $\Gamma_+ \cap \overline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) = \{w_1, w_2, \ldots\} = \Gamma_+ \cap \operatorname{supp}\mu$. A similar analysis for the other connected component of $\Gamma\backslash\{w\}$ finally gives $\Gamma \cap \overline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) = (\Gamma \cap \operatorname{supp}\mu) \cup \{w\}$.

Let us suppose now that $w \in \underline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) \backslash \operatorname{supp}\mu$. Let $\Gamma_- = (w_-, w)$ be the other connected component of $\Gamma\backslash\{w\}$, $\Gamma_-\backslash\operatorname{supp}\mu = (w_{-1}, w_0) \cup (w_{-2}, w_{-1}) \cup \cdots$ being the decomposition in connected components. From Theorem A and the condition for w we conclude that $(w_{-1}, w_1) \cap \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) = \{\lambda_0^{(n)}\}$ for n greater than certain index n_0, and $\lim_n \lambda_0^{(n)} = w$. From here on, similar arguments to the previous case prove that $\Gamma \cap \overline{\lim}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) = (\Gamma \cap \operatorname{supp}\mu) \cup \{w\}$. □

Theorem 4.11 says that, if we choose a sequence of para-orthogonal polynomials with a fixed zero w outside $\{\operatorname{supp}\mu\}'$, or with a zero converging to a point w outside $\operatorname{supp}\mu$, then, in the connected component of $\mathbb{T}\backslash\{\operatorname{supp}\mu\}'$ where w lies, the limit points of the zeros of the para-orthogonal polynomials coincide with $\operatorname{supp}\mu$ up to, at most, the point w.

Remark 4.12. The above result can be read as a statement about the zeros of the para-orthogonal polynomials $p_n^{u_n}(z) = z\varphi_{n-1}(z) + u_n\varphi_{n-1}^*(z)$. The requirement for fixing a common zero w for all these polynomials means that we have to choose $u_n = -w\varphi_{n-1}(w)/\varphi_{n-1}^*(w)$ for $n \geqslant 1$. However, if we consider Theorem 4.11 as a result about the eigenvalues of the unitary matrices $C(a_1, \ldots, a_{n-1}, u_n)$, the interesting question is: how to get from the sequence $\boldsymbol{a}$ of Schur parameters the sequence $\boldsymbol{u} = \boldsymbol{u}^w$ that fixes a common zero w for all the polynomials $p_n^{u_n}$? Since the polynomial $p_n^{u_n}$ is proportional to $q_n^{v_n} = \varphi_n^* - \overline{v}_n\varphi_n$ with $v_n = -u_n\frac{1-a_n\overline{u}_n}{1-\overline{a}_n u_n}$ (see Remark 2.3), we find that such a sequence $\boldsymbol{u} = \boldsymbol{u}^w$ must satisfy $u_{n+1}\overline{v}_n = -w$ and, thus,

$$
\begin{aligned}
u_1^w &= -w, \\
u_{n+1}^w &= w u_n^w \frac{1 - a_n\overline{u}_n^w}{1 - \overline{a}_n u_n^w}, \quad \forall n \geqslant 1.
\end{aligned} \tag{19}
$$

This recurrence answers the question.

Corollary 4.8 stated that, for the family of measures whose Schur parameters approach to the unit circle, it is possible to choose a sequence of para-orthogonal polynomials whose zeros exactly converge to the support of the measure. Theorem 4.11 gives another class of measures where this essentially happens, as the following corollary shows.

Corollary 4.13. *Let μ be a measure on $\mathbb{T}$ with a sequence $\boldsymbol{a}$ of Schur parameters, and let $\boldsymbol{u}$ be a sequence in $\mathbb{T}$. Assume that $\{\operatorname{supp}\mu\}'$ is connected.*

1. $w \in \Sigma_n(\boldsymbol{a};\boldsymbol{u})\backslash\{\operatorname{supp}\mu\}' \forall n \geqslant 1 \Rightarrow \lim_n \Sigma_n(\boldsymbol{a};\boldsymbol{u}) = \operatorname{supp}\mu \cup \{w\}$.
2. $w \in \underline{\lim}_n \Sigma_n(\boldsymbol{a};\boldsymbol{u})\backslash \operatorname{supp}\mu \Rightarrow \lim_n \Sigma_n(\boldsymbol{a};\boldsymbol{u}) = \operatorname{supp}\mu \cup \{w\}$.

The results of Section 3 provide very general situations where Theorem 4.11 can be applied. Concerning the more stringent result of Corollary 4.13, the following example gives a remarkable situation where it holds.

Example 4.14. The López class.
If $\lim_n \frac{a_{n+1}}{a_n} = \lambda \in \mathbb{T}$ and $\lim_n |a_n| = r \in (0,1)$, we know that $\{\operatorname{supp}\mu\}' = \triangle_\alpha(\lambda)$ with $\alpha \in (0,\pi)$ given by $\sin\frac{\alpha}{2} = r$. Therefore, for any $w \in \Gamma_\alpha(\lambda)$, $\lim_n \Sigma_n(\boldsymbol{a};\boldsymbol{u}^w) = \operatorname{supp}\mu \cup \{w\}$.

Theorem 4.11.2 has a consequence about the strong limit points of the zeros of para-orthogonal polynomials. It implies that $\underline{\lim}_n \Sigma_n(\boldsymbol{a};\boldsymbol{u})$ can differ from $\operatorname{supp}\mu$ in, at most, one point in each connected component of $\mathbb{T}\backslash\{\operatorname{supp}\mu\}'$. This result will be improved in Theorems 4.17, 4.18 and Corollary 4.19, which use some results of [21]. In this work, S. Khrushchev defines the so-called class of Markoff measures on the unit circle ($\operatorname{Mar}(\mathbb{T})$), which includes, as a particular case, all the measures whose support does not cover the unit circle [21, p. 268]. For a measure μ in this class he proves some results that give information about the asymptotics of φ_n/φ_n^* in $\mathbb{T}\backslash\operatorname{supp}\mu$, $(\varphi_n)_{n\geqslant 0}$ being the orthonormal polynomials in L^2_μ. As we will see, this is a key tool to control the strong limit points of the zeros of the para-orthogonal polynomials. Let us summarize the referred results in [21].

Theorem B (*Khrushchev [21, Lemma 8.4.1]*). *Let $\mu \in \operatorname{Mar}(\mathbb{T})$. Then there exists a positive number $\delta(\mu)$ such that*

$$\sup_{|z|\leqslant 1/2} \left|\frac{\varphi_n(z)}{\varphi_n^*(z)}\right| > \delta(\mu) > 0, \quad \forall n \geqslant 1.$$

Theorem C (*Khrushchev [21, Corollary 8.6]*). *Let B be a Blaschke product with zeros $\{z_j\}$ such that $|B(z_0)| > \delta > 0$ for some z_0, $|z_0| \leqslant 1/2$. If $|z - z_j| \geqslant \varepsilon > 0$, then $|B(z)| > c = c(\delta,\varepsilon) > 0$.*

The interest of this last result is that φ_n/φ_n^* is always a Blaschke product. With the above tools we get a result for the asymptotic behaviour of the sequence $(\varphi_n^*/\varphi_n)_{n\geqslant 0}$ on $\mathbb{C}\backslash\operatorname{Co}(\operatorname{supp}\mu)$, where $\operatorname{Co}(E)$ means the convex hull of $E \subset \mathbb{C}$.

Proposition 4.15. *Let μ be a measure on $\mathbb{T}$ and let $g_n := \varphi_n^*/\varphi_n$ for $n \geqslant 0$, $(\varphi_n)_{n\geqslant 0}$ being the orthonormal polynomials in L^2_μ. Then, any subsequence of $(g_n)_{n\geqslant 0}$ has a subsequence which uniformly converges on compact subsets of $\mathbb{C}\backslash\operatorname{Co}(\operatorname{supp}\mu)$.*

Proof. Let us suppose first that $\operatorname{supp}\mu \neq \mathbb{T}$, and let $\mathcal{K}$ be a compact subset of $\mathbb{C}\backslash\operatorname{Co}(\operatorname{supp}\mu)$. Since $\operatorname{Co}(\operatorname{supp}\mu)$ is also compact, the distance between $\mathcal{K}$ and $\operatorname{Co}(\operatorname{supp}\mu)$ must be a positive number ε. The zeros $\{z_j^{(n)}\}_{j=1}^n$ of the polynomial φ_n lie on $\operatorname{Co}(\operatorname{supp}\mu)$, so, $|z - z_j^{(n)}| \geqslant \varepsilon > 0$, $j = 1,\ldots,n$, for all $z \in \mathcal{K}$ and $n \in \mathbb{N}$. Therefore, Theorems B and C imply that $|1/g_n(z)| > c(\delta(\mu),\varepsilon) > 0$, $\forall z \in \mathcal{K}$, $\forall n \in \mathbb{N}$, and, thus, $(g_n)_{n\geqslant 0}$ is uniformly bounded on $\mathcal{K}$. That is, $(g_n)_{n\geqslant 0}$ is uniformly bounded on any compact subset of $\mathbb{C}\setminus\mathrm{Co}(\mathrm{supp}\,\mu)$. This is also true for the case $\mathrm{supp}\,\mu = \mathbb{T}$ since $|g_n(z)| \leqslant 1$ for $|z| \geqslant 1$. Therefore, $(g_n)_{n\geqslant 0}$ is always a normal family in $\mathbb{C}\setminus\mathrm{Co}(\mathrm{supp}\,\mu)$, which proves the proposition. □

Now we are ready to prove the main results about the strong limit points of the zeros of para-orthogonal polynomials. The following set will be important in the next discussions.

Definition 4.16. Given a sequence $\boldsymbol{E} = (E_n)_{n\geqslant 1}$, $E_n \subset \mathbb{C}$, we define $\overline{\overline{\lim}}_n\, E_n$ as the set of points $\lambda \in \mathbb{C}$ such that, for some infinite set $\mathcal{I} \subset \mathbb{N}$,

$$\lim_{n\in\mathcal{I}} d(\lambda, E_n) = \lim_{n\in\mathcal{I}} d(\lambda, E_{n+1}) = 0.$$

We call $\overline{\overline{\lim}}_n\, E_n$ the set of double limit points of the sequence $\boldsymbol{E}$.

Obviously, $\overline{\overline{\lim}}_n\, E_n$ is a closed set such that $\underline{\lim}_n\, E_n \subset \overline{\overline{\lim}}_n\, E_n \subset \overline{\lim}_n\, E_n$.

Theorem 4.17. *Let $\boldsymbol{a}$ be the sequence of Schur parameters of a measure μ on $\mathbb{T}$ and let $\boldsymbol{u}$ be a sequence in $\mathbb{T}$. Then, $\underline{\lim}_n\, \Sigma_n(\boldsymbol{a};\boldsymbol{u})$ coincides with* $\mathrm{supp}\,\mu$ *except, at most, at one point. If this point exists, $\overline{\overline{\lim}}_n\, \Sigma_n(\boldsymbol{a};\boldsymbol{u}) = \underline{\lim}_n\, \Sigma_n(\boldsymbol{a};\boldsymbol{u})$ and, thus, $\overline{\overline{\lim}}_n\, \Sigma_n(\boldsymbol{a};\boldsymbol{u})$ equals* $\mathrm{supp}\,\mu$ *up to such a point.*

Proof. It is enough to prove that the conditions $w \in \underline{\lim}_n\, \Sigma_n(\boldsymbol{a};\boldsymbol{u})\setminus\mathrm{supp}\,\mu$ and $z \in \overline{\overline{\lim}}_n\, \Sigma_n(\boldsymbol{a};\boldsymbol{u})\setminus\mathrm{supp}\,\mu$ imply $z = w$. Let $w \in \underline{\lim}_n\, \Sigma_n(\boldsymbol{a};\boldsymbol{u})$ and $z \in \overline{\overline{\lim}}_n\, \Sigma_n(\boldsymbol{a};\boldsymbol{u})$. There exist two sequences $(w_n)_{n\geqslant 1}$ and $(z_n)_{n\geqslant 1}$ with $w_n, z_n \in \Sigma_n(\boldsymbol{a};\boldsymbol{u})$, $\forall n \geqslant 1$, such that $\lim_n w_n = w$ and $\lim_{n\in\mathcal{I}} z_n = \lim_{n\in\mathcal{I}} z_{n+1} = z$ for some infinite set $\mathcal{I} \subset \mathbb{N}$. Let μ be the measure whose sequence of Schur parameters is $\boldsymbol{a}$, and let $g_n = \varphi_n^*/\varphi_n$ for $n \geqslant 0$, $(\varphi_n)_{n\geqslant 0}$ being the orthonormal polynomials in L^2_μ. Since w_n and z_n are zeros of the same para-orthogonal polynomial $p_n^{u_n}$, we get

$$\overline{z}_{n+1} g_n(z_{n+1}) = \overline{w}_{n+1} g_n(w_{n+1}), \quad n \geqslant 0. \tag{20}$$

Taking into account that $p_n^{u_n}$ is proportional to $q_n^{v_n} = \varphi_n^* - \overline{v}_n \varphi_n$ for some $v_n \in \mathbb{T}$ (see Remark 2.3), we also find that

$$g_n(z_n) = g_n(w_n), \quad n \geqslant 1. \tag{21}$$

Assume that $w, z \notin \mathrm{supp}\,\mu$ and let $\mathcal{K}$ be a compact subset of $\mathbb{T}\setminus\mathrm{supp}\,\mu$ containing two open arcs centred at w and z, respectively. $w_n, z_n \in \mathcal{K}$ for any n big enough. On the other hand, Proposition 4.15 ensures the uniform convergence on $\mathcal{K}$ of a subsequence $(g_n)_{n\in\mathcal{J}}$, $\mathcal{J} \subset \mathcal{I}$. If g is the uniform limit of this subsequence, $\lim_{n\in\mathcal{J}} g_n(w_n) = \lim_{n\in\mathcal{J}} g_n(w_{n+1}) = g(w)$ and $\lim_{n\in\mathcal{J}} g_n(z_n) = \lim_{n\in\mathcal{J}} g_n(z_{n+1}) = g(z)$. Taking limits for $n \in \mathcal{J}$ in (20) and (21) we conclude that $z = w$ since $g(z) \neq 0$ (in fact, $|g(z)| = 1$). □

It is clear that we can control the possible strong limit point of the zeros that lies outside the support of the measure choosing a sequence of para-orthogonal polynomials with a fixed zero outside this support. More surprising is that the choice of a fixed zero in the support always gives an exact equality between the strong limit points and the support of the measure. This is a consequence of the next theorem, which delimits the possible double limit points outside the support of the measure. In fact, it provides for any measure sequences of para-orthogonal polynomials that ensure the strict equality between the double limit points of the zeros and the support of the measure.

Theorem 4.18. *If* $\boldsymbol{a}$ *is the sequence of Schur parameters of a measure* μ *on* $\mathbb{T}$ *and* $\boldsymbol{u}$ *is a sequence in* $\mathbb{T}$,

$$\overline{\overline{\lim_n}}\, \Sigma_n(\boldsymbol{a};\boldsymbol{u})\backslash \operatorname{supp}\mu \subset \mathfrak{L}\left\{\frac{u_{n+1}}{u_n}\frac{1-\overline{a}_n u_n}{1-a_n\overline{u}_n}\right\}.$$

Proof. Let $z \in \overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a};\boldsymbol{u})$. There exists a sequence $(z_n)_{n\geqslant 1}$ such that $z_n \in \Sigma_n(\boldsymbol{a};\boldsymbol{u})$, $\forall n\geqslant 1$, and $\lim_{n\in\mathcal{I}} z_n = \lim_{n\in\mathcal{I}} z_{n+1} = z$ for some infinite set $\mathcal{I}\subset\mathbb{N}$. Since z_n is a zero of $p_n^{u_n}$ and $q_n^{v_n}$ with $v_n = -u_n\frac{1-a_n\overline{u}_n}{1-\overline{a}_n u_n}$ (see Remark 2.3), we find that $u_{n+1} = -z_{n+1}\overline{g_n(z_{n+1})}$ and $v_n = \overline{g_n(z_n)}$ for $n\geqslant 1$, which gives

$$\frac{u_{n+1}}{u_n}\frac{1-\overline{a}_n u_n}{1-a_n\overline{u}_n} = z_{n+1}\frac{g_n(z_n)}{g_n(z_{n+1})}, \quad n\geqslant 1. \tag{22}$$

Let us suppose that $z \notin \operatorname{supp}\mu$. Using again Proposition 4.15 we find that a subsequence $(g_n)_{n\in\mathcal{J}}$, $\mathcal{J}\subset\mathcal{I}$, uniformly converges to a function g on a compact subset of $\mathbb{T}\backslash\operatorname{supp}\mu$ containing an open arc centred at z. Hence, $\lim_{n\in\mathcal{J}} g_n(z_n) = \lim_{n\in\mathcal{J}} g_n(z_{n+1}) = g(z)$ and, taking limits for $n\in\mathcal{J}$ in (22), it follows that $z \in \mathfrak{L}\left\{\frac{u_{n+1}}{u_n}\frac{1-\overline{a}_n u_n}{1-a_n\overline{u}_n}\right\}$. □

As a first consequence of the previous theorem we find infinitely many sequences of para-orthogonal polynomials $(p_n^{u_n})_{n\geqslant 1}$, the double limit points of whose zeros coincide exactly with $\operatorname{supp}\mu$. They are those defined by sequences $\boldsymbol{u}$ such that $\mathfrak{L}\left\{\frac{u_{n+1}}{u_n}\frac{1-\overline{a}_n u_n}{1-a_n\overline{u}_n}\right\} \subset \operatorname{supp}\mu$.

An interesting choice for $\boldsymbol{u}$ is given by the phases of $\boldsymbol{a}$, that is, $u_n = \frac{a_n}{|a_n|}$ if $a_n \neq 0$ and u_n arbitrarily chosen in $\mathbb{T}$ otherwise. Then, the previous theorem states that $\overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a};\boldsymbol{u})\backslash\operatorname{supp}\mu \subset \mathfrak{L}\{\overline{u}_n u_{n+1}\}$.

If we are interested in locating the possible strong limit point outside $\operatorname{supp}\mu$ at a certain place $w\in\mathbb{T}$, we can choose $\boldsymbol{u}$ so that $\lim_n \frac{u_{n+1}}{u_n}\frac{1-\overline{a}_n u_n}{1-a_n\overline{u}_n} = w$. Then, $\overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a};\boldsymbol{u}) \subset \operatorname{supp}\mu\cup\{w\}$. So, if $w\in\operatorname{supp}\mu$, $\overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a};\boldsymbol{u}) = \operatorname{supp}\mu$.

A particular choice of $\boldsymbol{u}$ which ensures $\overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a};\boldsymbol{u}) \subset \operatorname{supp}\mu\cup\{w\}$ is given by the recurrence

$$u_{n+1} = wu_n\frac{1-a_n\overline{u}_n}{1-\overline{a}_n u_n}, \quad n\geqslant 1 \tag{23}$$

with an initial condition $u_1 = u$, u an arbitrary point in $\mathbb{T}$. Let us study the form of the related para-orthogonal polynomials $p_n^{u_n}$.

Without loss of generality we write $u_n = -wr_{n-1}(w)/r_{n-1}^*(w)$, with r_n a polynomial of degree n. Eq. (23) is equivalent to $r_n(w)/r_n^*(w) = s_n(w)/s_n^*(w)$, $s_n(w) = wr_{n-1}(w) + a_n r_{n-1}^*(w)$. So, $r_n(w) = \lambda_n(wr_{n-1}(w) + a_n r_{n-1}^*(w))$, $\lambda_n \in \mathbb{R}\backslash\{0\}$. This equation has two independent solutions: $(\delta_n\varphi_n(w))_{n\geqslant 0}$, $(i\delta_n\psi_n(w))_{n\geqslant 0}$, where $\delta_n = \lambda_1\rho_1\cdots\lambda_n\rho_n$, $(\varphi_n)_{n\geqslant 0}$ are the orthonormal polynomials related to the Schur parameters $\boldsymbol{a}$, and $(\psi_n)_{n\geqslant 0}$ are the orthonormal second kind polynomials, associated with the Schur parameters $-\boldsymbol{a}$ [28,12].

Therefore, the sequence $\boldsymbol{u}$ satisfies (23) if and only if $p_n^{u_n}(z)$ is proportional to $p_{n-1}^*(w)z\varphi_{n-1}(z) - wp_{n-1}(w)\varphi_{n-1}^*(z)$, where $p_n = c_1\varphi_n + ic_2\psi_n$, $(c_1,c_2)\in\mathbb{R}^2\backslash\{(0,0)\}$. Since $\rho_n p_n(w) =$

$wp_{n-1}(w) + a_n p^*_{n-1}(w)$, this is equivalent to saying that $p_n^{u_n}(z)$ is proportional to $p^*_n(w)\varphi_n(z) - p_n(w)\varphi^*_n(z)$. Notice that the initial condition $u = -w$ means that p_0 is real, which corresponds to the case $c_2 = 0$, giving the sequence of para-orthogonal polynomials with a fixed zero at w studied in Remark 4.12.

Summarizing, as a consequence of Theorem 4.18 we obtain the following result.

Corollary 4.19. *Let μ be a measure on $\mathbb{T}$, $(\varphi_n)_{n\geqslant 0}$ be the orthonormal polynomials in L^2_μ and $(\psi_n)_{n\geqslant 0}$ be the related orthonormal second kind polynomials. Any sequence $(P_n)_{n\geqslant 1}$ of para-orthogonal polynomials given by*

$$P_n(z) := p^*_n(w)\varphi_n(z) - p_n(w)\varphi^*_n(z),$$

$$p_n := c_1\varphi_n + ic_2\psi_n, \quad (c_1, c_2) \in \mathbb{R}^2\backslash\{(0,0)\}, \quad w \in \mathbb{T},$$

has the property that the double limit points of the corresponding zeros coincide with $\operatorname{supp}\mu$ *except, at most, at the point w. Hence, if $w \in \operatorname{supp}\mu$, the double limit points of the zeros exactly coincide with* $\operatorname{supp}\mu$. *When $c_2 = 0$, all the para-orthogonal polynomials have a common zero at w and the double and strong limit points coincide with* $\operatorname{supp}\mu \cup \{w\}$.

The para-orthogonal polynomials $(P_n)_{n\geqslant 1}$ given in the previous corollary appeared previously in [6], where it was proved that they have other interesting properties concerning the interlacing of zeros: for all n, P_n and P_{n+1} have interlacing zeros in $\mathbb{T}\backslash\{w\}$ [6, Theorem 1].

In [14], L. Golinskii conjectured that the strong limit points (which he called the strong attracting points) of the zeros of para-orthogonal polynomials with a fixed zero lying on the support of the measure, must coincide with this support. Theorems 4.17, 4.18 and Corollary 4.19, not only confirm this conjecture, but go even further in two senses: the achieved results cover any sequence of para-orthogonal polynomials, not only the case of a fixed zero on the support of the measure; even in this case, the results are stronger than the one conjectured by L. Golinskii since we have proved the equality between the support of the measure and the double limit points of the zeros of the para-orthogonal polynomials. Concerning related results for orthogonal polynomials on the real line see [9].

The previous results give a method for approximating the support of a measure μ on the unit circle starting from its sequence $\boldsymbol{a}$ of Schur parameters, based on the computation of the eigenvalues of the finite unitary matrices $C(a_1, \ldots, a_{n-1}, u_n)$ for a sequence $\boldsymbol{u}$ in $\mathbb{T}$. A recommendable choice is the sequence $\boldsymbol{u} = \boldsymbol{u}^w$ given in (19) that fixes a common eigenvalue w for all the finite matrices, because it permits us to control the only possible strong limit point that, according to Theorem 4.17, can lie outside $\operatorname{supp}\mu$. In this case Theorem 4.18 proves that the double limit points coincide with $\operatorname{supp}\mu \cup \{w\}$, so the computation of the eigenvalues for pairs of consecutive matrices can be used to eliminate those weak limit points that are spurious points of $\operatorname{supp}\mu$.

The following figures show some examples of the previous method of approximation. They represent $\Sigma_n(\boldsymbol{a}; \boldsymbol{u})$ for some choices of $\boldsymbol{a}$, $\boldsymbol{u}$ and $n = 1000, 1001$. The computations have been made applying the double precision routines of MATLAB to the calculation of the eigenvalues of $C(a_1, \ldots, a_{n-1}, u_n)$. We have to remark that the computations can be also made using a Hessenberg matrix unitarily equivalent to $C(a_1, \ldots, a_{n-1}, u_n)$ [1,8]. However, although the time of computation of eigenvalues is only a little smaller with the five-diagonal representation (using the standard routines), the computational cost of building the matrix is much bigger in the Hessenberg case, growing very much faster as n increases.

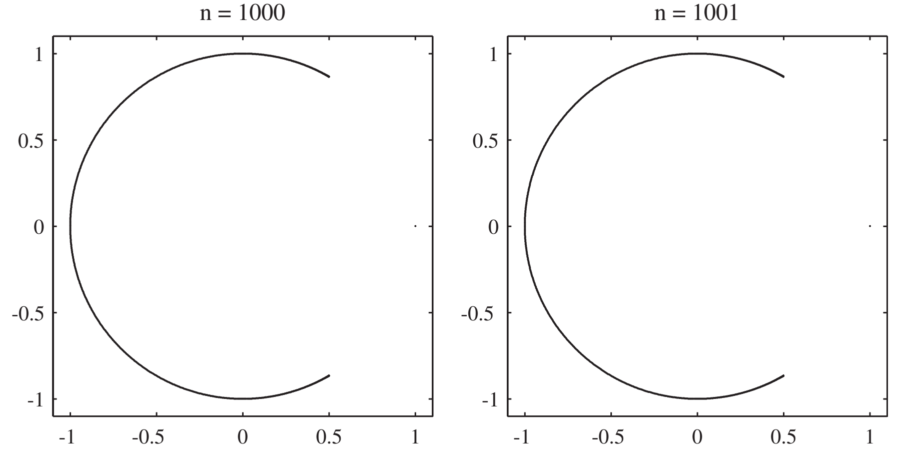

Fig. 1. $\Sigma_n(\boldsymbol{a}; \boldsymbol{u}^1)$ for $a_n = \frac{1}{2}$.

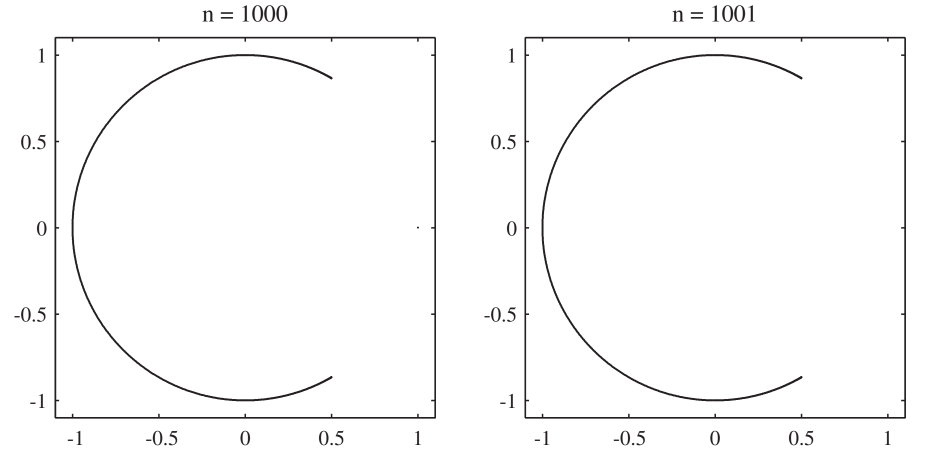

Fig. 2. $\Sigma_n(\boldsymbol{a}; \boldsymbol{u}^{-1})$ for $a_n = \frac{1}{2}$.

The first three figures correspond to different choices of $\boldsymbol{u}$ in the case of constant Schur parameters $a_n = \frac{1}{2}$, where $\operatorname{supp} \mu = \triangle_{\frac{\pi}{3}}(1)$. Corollary 4.13 proves that $\lim_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}^1) = \operatorname{supp} \mu \cup \{1\}$, as can be seen in Fig. 1. According to Corollary 4.19, $\overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}^{-1}) = \operatorname{supp} \mu$. This is in agreement with Fig. 2, where we see that $\lim_n \Sigma_{2n-1}(\boldsymbol{a}; \boldsymbol{u}^{-1}) = \operatorname{supp} \mu$ but $\lim_n \Sigma_{2n}(\boldsymbol{a}; \boldsymbol{u}^{-1}) = \operatorname{supp} \mu \cup \{1\}$, so, 1 is a weak but not a double limit point. Such behaviour was predicted by L. Golinskii in [14, Example 10]. As we have seen throughout the paper, another interesting choice is $u_n = a_n/|a_n|$ which, used in Fig. 3, seems to give $\lim_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}) = \operatorname{supp} \mu$, although Theorem 4.18 says that $\overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u})$ could differ from $\operatorname{supp} \mu$ in (at most) the point 1.

The next three figures deal with the 2-periodic Schur parameters $a_{2n-1} = \frac{1}{4}$, $a_{2n} = \frac{3}{4}$, whose measure satisfies $\{\operatorname{supp} \mu\}' = \triangle_{\alpha_+}(1) \cap \triangle_{\alpha_-}(-1)$ with $\alpha_+ \approx 0.35\pi$ and $\alpha_- \approx 0.19\pi$. According to those figures there are no isolated mass points. From Corollary 4.19, $\overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a}; \boldsymbol{u}^1) = \operatorname{supp} \mu \cup$

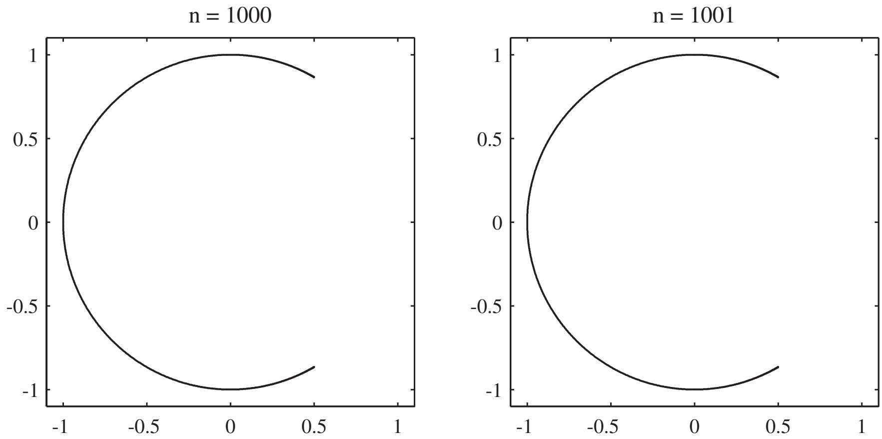

Fig. 3. $\Sigma_n(\boldsymbol{a};\boldsymbol{u})$ for $a_n = \frac{1}{2}$ and $u_n = \frac{a_n}{|a_n|}$.

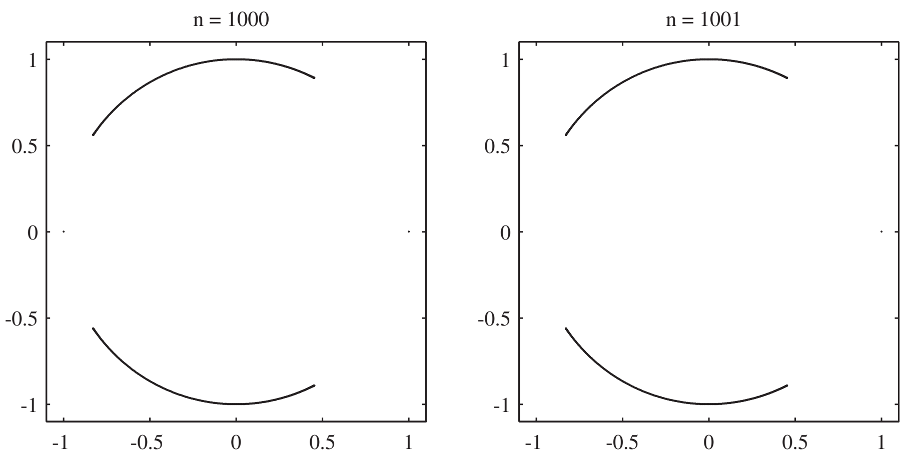

Fig. 4. $\Sigma_n(\boldsymbol{a};\boldsymbol{u}^1)$ for $a_{2n-1} = \frac{1}{4}$ and $a_{2n} = \frac{3}{4}$.

$\{1\}$, while Theorem 4.11 states that, apart from the point 1, there cannot be weak limit points in the gap around 1. These results agree with Fig. 4 which shows that, in this case, -1 is the only weak limit point that is not a double limit point. The choice $\boldsymbol{u} = \boldsymbol{u}^i$ fixes a common eigenvalue at $\operatorname{supp}\mu$ and, thus, Corollary 4.19 implies that $\overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a};\boldsymbol{u}^i) = \operatorname{supp}\mu$. This is the situation represented in Fig. 5. Fig. 6 seems to indicate that, as in the case $a_n = \frac{1}{2}$, $\overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a};\boldsymbol{u}) = \operatorname{supp}\mu$ also for $u_n = a_n/|a_n|$, where Theorem 4.18 only predicts that $\overline{\overline{\lim}}_n \Sigma_n(\boldsymbol{a};\boldsymbol{u}) \subset \operatorname{supp}\mu \cup \{1\}$. However, contrary to the case of constant Schur parameters, a weak limit point, -1, appears now outside $\operatorname{supp}\mu$.

It is interesting to compare the first example, $a_n = \frac{1}{2}$, with the situation of a random sequence $\boldsymbol{a}$ lying on $\operatorname{Re}(z) \geqslant \frac{1}{2}$. Fig. 7, which represents this last case for $\boldsymbol{u} = \boldsymbol{u}^{-1}$, agrees with Corollary 3.5, which predicts that $\{\operatorname{supp}\mu\}' \subset \triangle_\alpha(1)$, $\alpha \approx 0.24\pi$.

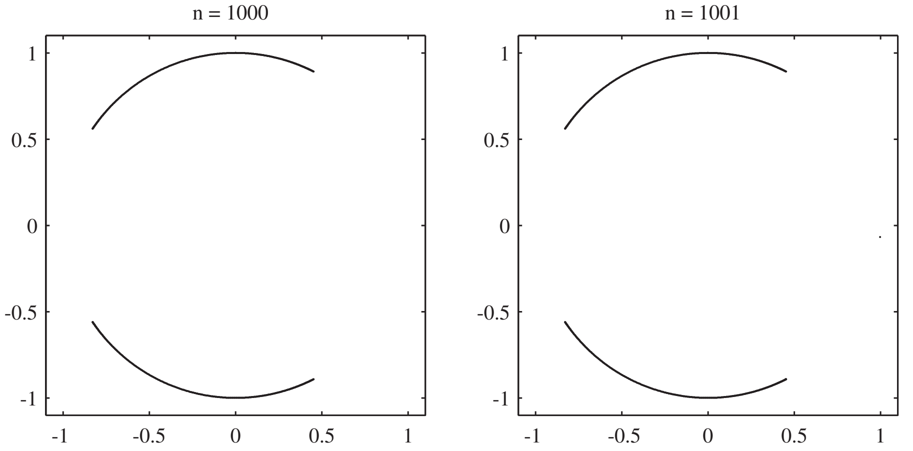

Fig. 5. $\Sigma_n(\boldsymbol{a};\boldsymbol{u}^i)$ for $a_{2n-1}=\frac{1}{4}$ and $a_{2n}=\frac{3}{4}$.

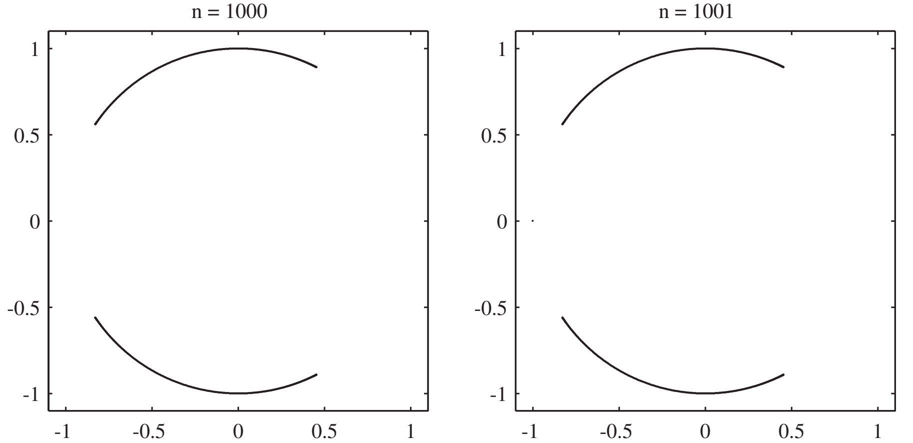

Fig. 6. $\Sigma_n(\boldsymbol{a};\boldsymbol{u})$ for $a_{2n-1}=\frac{1}{4}$, $a_{2n}=\frac{3}{4}$ and $u_n=\frac{a_n}{|a_n|}$.

The second example of 2-periodic Schur parameters, $a_{2n-1}=\frac{1}{4}$, $a_{2n}=\frac{3}{4}$, can be compared with Figs. 8 and 9. Fig. 8 deals with the choice $\boldsymbol{u}=\boldsymbol{u}^1$ for a sequence $\boldsymbol{a}$ whose odd and even subsequences are randomly located on $\operatorname{Re}(z)\leqslant\frac{1}{4}$ and $\operatorname{Re}(z)\geqslant\frac{3}{4}$, respectively. This figure confirms Theorem 3.4, which implies that $\{\operatorname{supp}\mu\}'\subset\triangle_\alpha(-1)$, $\alpha=\frac{\pi}{6}$. In Fig. 9 the odd subsequence of $\boldsymbol{a}$ is randomly chosen on the semicircle $\overline{\Gamma}_{\frac{\pi}{2}}(\frac{1}{4})$ and $a_{2n}=\frac{3}{4}$ with $\boldsymbol{u}=\boldsymbol{u}^i$. The figure is compatible with Theorems 3.16 and 3.4 which give $\{\operatorname{supp}\mu\}'\subset\triangle_\alpha(-1)\cap\triangle_\beta(1)$, $\alpha=\frac{\pi}{6}$, $\beta\approx 0.24\pi$.

Finally, Figs. 10–12 correspond to different sequences of Schur parameters having two different limit points $re^{\pm i\frac{\pi}{3}}$ with equal modulus $r=\sin(\frac{3\pi}{8})$. In Figs. 10 and 11, the subsequences of $\boldsymbol{a}$ converging to such limit points are chosen so that $(a_{n+1}/a_n)_{n\geqslant 1}$ has three limit points $1, e^{\pm i\frac{2\pi}{3}}$. Then, Theorem 3.10 predicts that $\{\operatorname{supp}\mu\}'$ is included in three arcs centred at $-1, e^{\pm i\frac{\pi}{3}}$ with

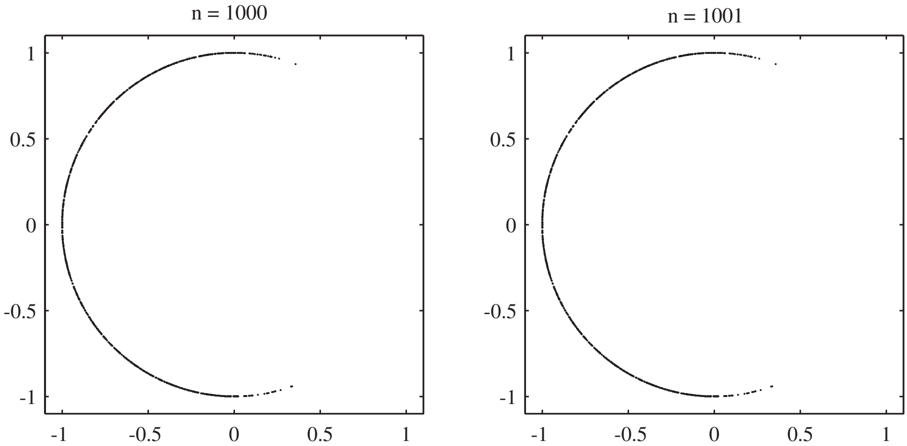

Fig. 7. $\Sigma_n(\boldsymbol{a}; \boldsymbol{u}^{-1})$ for a_n randomly distributed on $\mathrm{Re}(z) \geqslant \frac{1}{2}$.

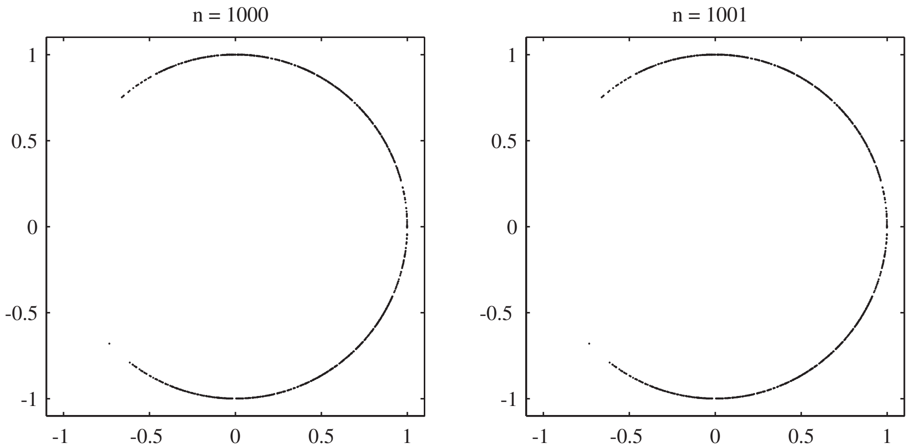

Fig. 8. $\Sigma_n(\boldsymbol{a}; \boldsymbol{u}^{1})$ for a_{2n-1} and a_{2n} randomly distributed on $\mathrm{Re}(z) \leqslant \frac{1}{4}$ and $\mathrm{Re}(z) \geqslant \frac{3}{4}$, respectively.

angular radius $\frac{\pi}{4}$. As Example 3.12 shows, this means that $\{\mathrm{supp}\,\mu\}'$ has at least three gaps centred at 1 and $e^{\pm i\frac{2\pi}{3}}$ with an angular radius greater than or equal to $\alpha = \frac{\pi}{12}$. In fact, from Example 3.7 we see that Corollary 3.5 ensures that the radius of the gap around 1 is not less than $\beta \approx 0.22\pi$ and, hence, $\{\mathrm{supp}\,\mu\}' \subset \triangle_\alpha(e^{i\frac{2\pi}{3}}) \cap \triangle_\alpha(e^{-i\frac{2\pi}{3}}) \cap \triangle_\beta(1)$. This result agrees with Figs. 10 and 11. The comparison with Fig. 12 is of interest. It represents the case of 2-periodic Schur parameters with the same limit points as in Figs. 10 and 11. In this case $(a_{n+1}/a_n)_{n\geqslant 1}$ has only two limit points $e^{\pm i\frac{2\pi}{3}}$, hence, the arc around -1 is now free of $\{\mathrm{supp}\,\mu\}'$. In fact, we know that $\{\mathrm{supp}\,\mu\}' = \triangle_{\alpha_-}(-1) \cap \triangle_{\alpha_+}(1)$ with $\alpha_- \approx 0.59\pi$ and $\alpha_+ \approx 0.31\pi$. Notice also the similarity between Figs. 10–12 concerning the isolated mass point close to $e^{i\frac{2\pi}{3}}$.

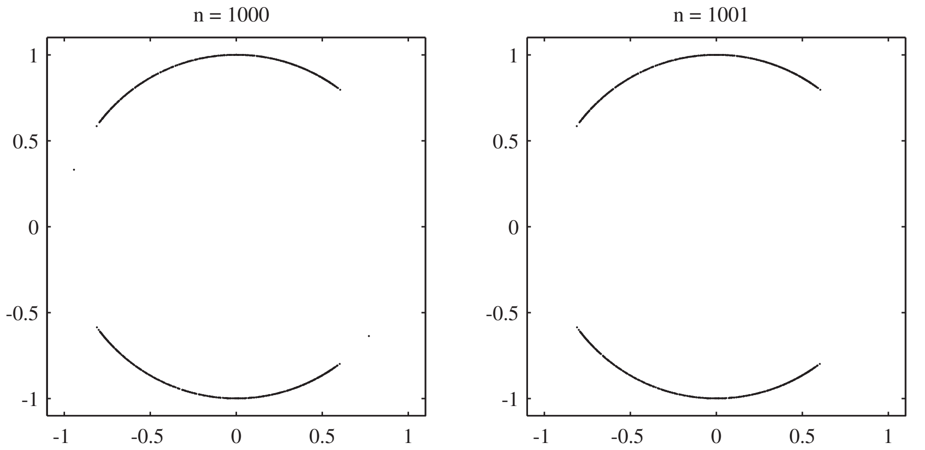

Fig. 9. $\Sigma_n(\boldsymbol{a}; \boldsymbol{u}^i)$ for a_{2n-1} randomly distributed on $\overline{\Gamma}_{\frac{\pi}{2}}(\frac{1}{4})$ and $a_{2n} = \frac{3}{4}$.

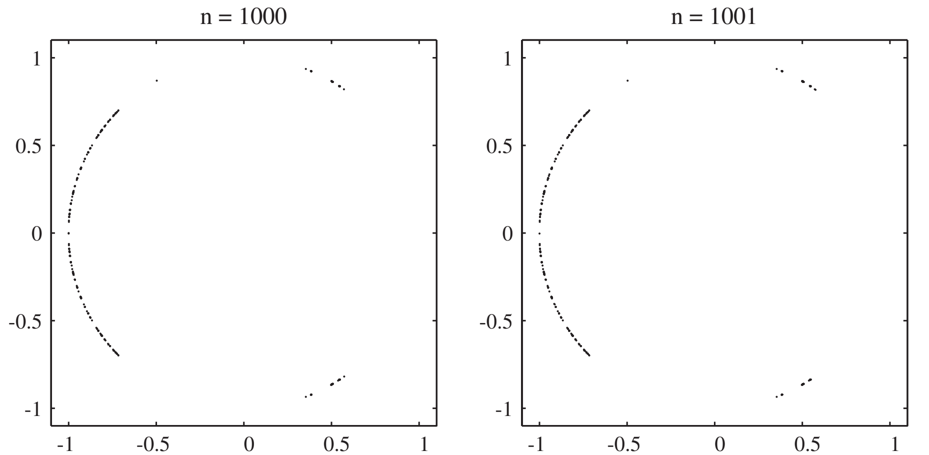

Fig. 10. $\Sigma_n(\boldsymbol{a}; \boldsymbol{u}^w)$ for $a_n = \sin(\frac{3\pi}{8})e^{\pm i\frac{\pi}{3}}$ if n is prime/not prime and $w = e^{i\frac{\pi}{3}}$.

5. Applications to the continued fractions

In this section, we will show some applications of the previous results to the study of rational approximants of Carathéodory functions. In what follows, $f_n(z) \rightrightarrows f(z)$, $z \in \Omega$, means that the sequence $(f_n)_{n \geqslant 1}$ uniformly converges to f on compact subsets of Ω.

It is known that the monic orthogonal polynomials $(\Phi_n)_{n \geqslant 0}$ corresponding to a measure μ on $\mathbb{T}$ and the related monic second kind polynomials $(\Psi_n)_{n \geqslant 0}$ provide rational approximants for the associated Carathéodory function

$$F_\mu(z) := \int_{\mathbb{T}} \frac{\lambda + z}{\lambda - z} \, d\mu(\lambda).$$

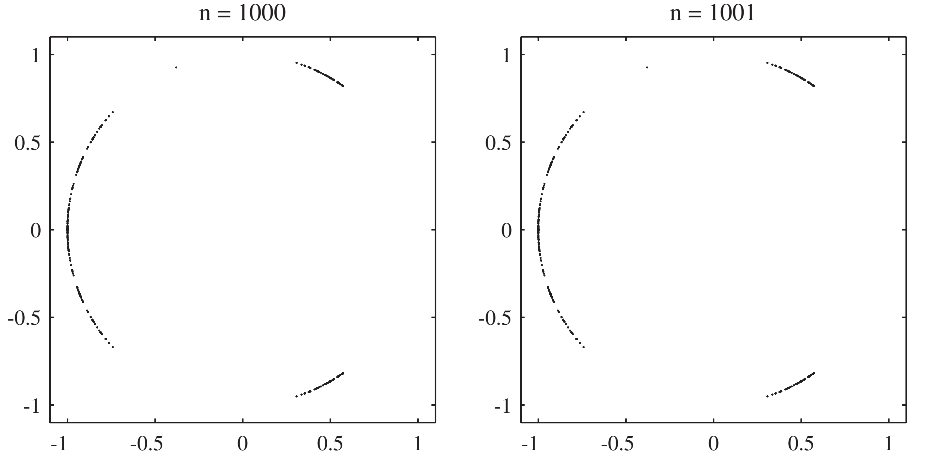

Fig. 11. $\Sigma_n(\boldsymbol{a}; \boldsymbol{u}^w)$ for a_n randomly distributed on $\{\sin(\frac{3\pi}{8})e^{\pm i\frac{\pi}{3}}\}$ and $w = e^{i\frac{\pi}{3}}$.

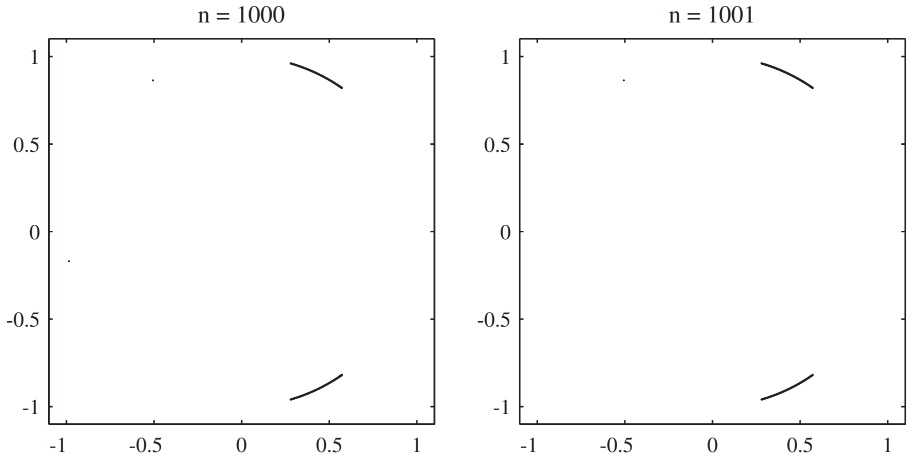

Fig. 12. $\Sigma_n(\boldsymbol{a}; \boldsymbol{u}^w)$ for $a_n = \sin(\frac{3\pi}{8})e^{\pm i\frac{\pi}{3}}$ if n is even/odd and $w = e^{i\frac{\pi}{3}}$.

More precisely, $\Psi_n^*(z)/\Phi_n^*(z) \rightrightarrows F_\mu(z)$, $z \in \mathbb{D}$, and $-\Psi_n(z)/\Phi_n(z) \rightrightarrows F_\mu(z)$, $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$ [20]. When supp $\mu \neq \mathbb{T}$, it is possible to enlarge the above domains of convergence through a careful analysis of the asymptotic behaviour of the zeros of the orthonormal polynomials [21, Section 9].

All these can be read as results about the convergence of continued fractions. Remember that, given a continued fraction

$$K := \beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \ddots}},$$

the related w-modified nth approximant is

$$K_n^w := \beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \ddots + \cfrac{\alpha_n}{\beta_n + w}}}.$$

In particular, $K_n := K_n^0$ is called the nth approximant of K. It is known that (see [30])

$$K_n^w = \frac{A_n + wA_{n-1}}{B_n + wB_{n-1}}, \quad n \geqslant 0, \tag{24}$$

where A_n and B_n are given by the same recurrence

$$X_n = \beta_n X_{n-1} + \alpha_n X_{n-2}, \quad X_n = A_n, B_n, \quad n \geqslant 0,$$

but with different initial conditions $A_0 = \beta_0$, $A_{-1} = 1$ and $B_0 = 1$, $B_{-1} = 0$.

The recurrences for Φ_n and Ψ_n show that $A_{2n} = \Psi_n^*(z)$, $B_{2n} = \Phi_n^*(z)$, $A_{2n+1} = -z\Psi_n(z)$ and $B_{2n+1} = z\Phi_n(z)$ for the continued fraction [20]

$$K(\boldsymbol{a}; z) := 1 + \cfrac{-2z}{z + \cfrac{1}{\overline{a}_1 + \cfrac{\rho_1^2 z}{a_1 z + \cfrac{1}{\overline{a}_2 + \cfrac{\rho_2^2 z}{a_2 z + \ddots}}}}}, \tag{25}$$

where $\boldsymbol{a}$ is the sequence of Schur parameters of μ. Hence, $\Psi_n^*(z)/\Phi_n^*(z)$ and $-\Psi_n(z)/\Phi_n(z)$ are, respectively, the $2n$th approximant $K_{2n}(a_1, \ldots, a_n; z)$ and the $2n+1$th approximant $K_{2n+1}(a_1, \ldots, a_n; z)$ of $K(\boldsymbol{a}; z)$.

It is also clear from (24) that for any $u \in \mathbb{T}$ [20]

$$K_{2n}(a_1, \ldots, a_{n-1}, u; z) = K_{2n-1}^u(a_1, \ldots, a_{n-1}; z) = -\frac{\Psi_n^u(z)}{\Phi_n^u(z)}, \quad n \geqslant 1, \tag{26}$$

where $\Phi_n^u(z) := z\Phi_{n-1}(z) + u\Phi_{n-1}^*(z)$ and $\Psi_n^u(z) := z\Psi_{n-1}(z) - u\Psi_{n-1}^*(z)$. Therefore, these modified approximants are quotients of para-orthogonal polynomials. In fact, given a sequence $\boldsymbol{u}$ in $\mathbb{T}$, the convergence properties for the modified approximants $-\Psi_n^{u_n}/\Phi_n^{u_n}$ are, in general, better than for the standard ones, since it is known that $-\Psi_n^{u_n}(z)/\Phi_n^{u_n}(z) \rightrightarrows F_\mu(z)$, $z \in \mathbb{C}\backslash\mathbb{T}$ [20]. The aim of this section is to find information about the convergence of these modified approximants on the unit circle.

Closely related to the concept of Carathéodory function is the notion of resolvent $R_z(T) := (z - T)^{-1}$ of an operator $T \in \mathfrak{B}(H)$, which is again a bounded operator on H for $z \in \mathbb{C}\backslash\sigma(T)$. Moreover, when T is normal, $\|R_z(T)\| = 1/d(z, \sigma(T))$ for $z \in \mathbb{C}\backslash\sigma(T)$. The Carathéodory function of a measure μ on $\mathbb{T}$ with Schur parameters sequence $\boldsymbol{a}$ is related to the resolvent $R_z(\boldsymbol{a}) := R_z(C(\boldsymbol{a}))$, which is a bounded operator on ℓ^2 for $z \in \mathbb{C}\backslash\operatorname{supp}\mu$. In fact,

$$\int_{\mathbb{T}} \lambda^n \, d(E_{C(\boldsymbol{a})}(\lambda)e_1, e_1) = (C(\boldsymbol{a})^n e_1, e_1) = (U^{\mu n}1, 1) = \int_{\mathbb{T}} \lambda^n \, d\mu(\lambda), \quad \forall n \in \mathbb{Z},$$

and, thus, $d\mu(\lambda) = d(E_{C(\boldsymbol{a})}(\lambda)e_1, e_1)$. Therefore,

$$F_\mu(z) = \int_{\mathbb{T}} \frac{\lambda + z}{\lambda - z}\, d(E_{C(\boldsymbol{a})}(\lambda)e_1, e_1) = 1 - 2z(R_z(\boldsymbol{a})e_1, e_1).$$

Also, for any $u \in \mathbb{T}$, the modified approximant $-\Psi_n^u/\Phi_n^u$ is related to the resolvent $R_z(a_1, \ldots, a_{n-1}, u) := R_z(C(a_1, \ldots, a_{n-1}, u))$, which defines an operator on ℓ_n^2 for z outside the spectrum of $C(a_1, \ldots, a_{n-1}, u)$. More precisely, if $f_j := 1-2z(R_z(a_1, \ldots, a_{n-1}, u)e_1, e_j)$ for $j = 1, \ldots, n$, the vector $\boldsymbol{f} := \sum_{j=1}^n f_j e_j$ satisfies

$$(C(a_1, \ldots, a_{n-1}, u) - z)\boldsymbol{f} = (C(a_1, \ldots, a_{n-1}, u) + z)e_1.$$

Just solving this system for f_1 we get

$$f_1 = -\frac{\det(z - VC(-a_1, \ldots, -a_{n-1}, -u)V^*)}{\det(z - C(a_1, \ldots, a_{n-1}, u))},$$

where V is the linear operator on ℓ_n^2 defined by $Ve_j = (-1)^j e_j$, $j = 1, \ldots, n$. From Corollary 2.4, $\Phi_n^u(z) = \det(z - C(a_1, \ldots, a_{n-1}, u))$, so, we finally get

$$-\frac{\Psi_n^u(z)}{\Phi_n^u(z)} = 1 - 2z(R_z(a_1, \ldots, a_{n-1}, u)e_1, e_1).$$

As a consequence of the previous discussion, given a sequence $\boldsymbol{u}$ in $\mathbb{T}$, the weak convergence of $(\hat{R}_z(a_1, \ldots, a_{n-1}, u_n))_{n\geqslant 1}$ to $R_z(\boldsymbol{a})$ implies the convergence of $(-\Psi_n^{u_n}(z)/\Phi_n^{u_n}(z))_{n\geqslant 1}$ to $F_\mu(z)$. In the case of self-adjoint band operators, the convergence of the resolvents of finite orthogonal truncations was analysed in [4] and [18], in connection with its interest for the Jacobi fractions. An extension of the ideas in [4] and [18] gives the following result.

Proposition 5.1. *Let $T \in \mathfrak{B}(\ell^2)$ be a normal band operator. If T_n is a normal truncation of T on ℓ_n^2 for $n \geqslant 1$ and $(\|T_n\|)_{n\geqslant 1}$ is bounded, for all $x \in \ell^2$,*

$$\hat{R}_z(T_n)x \rightrightarrows R_z(T)x, \quad z \in \mathbb{C}\setminus \overline{\lim_n}\, \sigma(T_n).$$

Moreover, each $z \in \overline{\lim}_n\, \sigma(T_n) \setminus \underline{\lim}_n\, \sigma(T_n)$ has a neighbourhood where the above uniform convergence holds at least for a subsequence of $(T_n)_{n\geqslant 1}$.

Proof. Let $z \in \mathbb{C}\setminus \underline{\lim}_n\, \sigma(T_n)$. There exist $\delta > 0$ and a subsequence $(T_n)_{n\in\mathcal{I}}$ such that $d(z, \sigma(T_n)) \geqslant \delta$, $\forall n \in \mathcal{I}$. Hence, $D_\delta(z) \subset \mathbb{C}\setminus \underline{\lim}_n\, \sigma(T_n)$ and, from Proposition 2.1.1, $D_\delta(z) \subset \mathbb{C}\setminus\sigma(T)$. Therefore, $R_w(T) \in \mathfrak{B}(\ell^2)$ for $w \in D_\delta(z)$. Also, $R_w(T_n)$ exists for $n \in \mathcal{I}$ and $w \in D_\delta(z)$. Moreover, since T_n is normal, $\|R_w(T_n)\| = 1/d(w, \sigma(T_n)) \leqslant 1/(\delta - |w - z|)$ for $n \in \mathcal{I}$. Hence, $(\|R_w(T_n)\|)_{n\in\mathcal{I}}$ is uniformly bounded with respect to w on compact subsets of $D_\delta(z)$.

Let P_n be the orthogonal projection on ℓ_n^2 and $w \in D_\delta(z)$. From the identities $\hat{R}_w(T_n) = P_n R_w(\hat{T}_n)$ and $R_w(\hat{T}_n) - R_w(T) = R_w(\hat{T}_n)(\hat{T}_n - T)R_w(T)$ we get

$$\hat{R}_w(T_n) - R_w(T) = \hat{R}_w(T_n)(\hat{T}_n - T)R_w(T) + (P_n - 1)R_w(T).$$

Proposition 2.1.1 states that $\hat{T}_n \to T$. Since $P_n \to 1$ and $(\|R_w(T_n)\|)_{n\in\mathcal{I}}$ is bounded we conclude that $\hat{R}_w(T_n) \underset{n\in\mathcal{I}}{\to} R_w(T)$. Moreover, the equality

$$\hat{R}_{w'}(T_n) - R_{w'}(T) = \hat{R}_w(T_n) - R_w(T) + (w' - w)(R_w(T)R_{w'}(T) - \hat{R}_{w'}(T_n)\hat{R}_w(T_n))$$

shows that, given $x \in H$ and $\varepsilon > 0$, there is a disk centred at w such that $\|\hat{R}_{w'}(T_n)x - R_{w'}(T)x\| < \varepsilon$ for w' lying on such a disk and n big enough. Then, standard arguments prove that $\hat{R}_w(T_n)x \underset{n\in\mathcal{I}}{\rightrightarrows} R_w(T)x$, $w \in D_\delta(z)$.

In the preceding discussion, if $z \notin \overline{\lim}_n\, \sigma(T_n)$, the subsequence $(T_n)_{n\in\mathcal{I}}$ can be chosen such that $\mathcal{I} = \{n \in \mathbb{N} : n \geqslant N\}$, $N \in \mathbb{N}$, and, so, the uniform convergence $\hat{R}_w(T_n)x \rightrightarrows R_w(T)x$, $w \in D_\delta(z)$, holds for the full sequence. Therefore, the convergence is uniform on compact subsets of $\mathbb{C}\setminus \overline{\lim}_n\, \sigma(T_n)$. □

From the preceding proposition a result for the resolvent of $C(\boldsymbol{a})$ immediately follows.

Theorem 5.2. *Given a sequence* $\boldsymbol{a}$ *in* $\mathbb{D}$ *and a sequence* $\boldsymbol{u}$ *in* $\mathbb{T}$, *for all* $x \in \ell^2$,

$$\hat{R}_z(a_1, \ldots, a_{n-1}, u_n)x \rightrightarrows R_z(\boldsymbol{a})x, \quad z \in \mathbb{C}\setminus \overline{\lim_n}\, \Sigma_n(\boldsymbol{a}; \boldsymbol{u}).$$

Moreover, each $z \in \overline{\lim}_n\, \Sigma_n(\boldsymbol{a}; \boldsymbol{u})\setminus \operatorname{supp}\mu$ (*up to, at most, one point*) *has a neighbourhood where the above uniform convergence holds at least for a subsequence.*

Proof. Apply Proposition 5.2 to $C(\boldsymbol{a})$ and its finite unitary truncations $C(a_1, \ldots, a_n, u_n)$, taking into account that, from Theorem 4.17, $\underline{\lim}_n\, \Sigma_n(\boldsymbol{a}; \boldsymbol{u})$ coincides with $\operatorname{supp}\mu$ up to, at most, at one point. □

Since the strong convergence of operators implies the weak convergence, we get a conclusion for the convergence of the modified approximants (26).

Corollary 5.3. *If* $\boldsymbol{a}$ *is the sequence of Schur parameters of a measure* μ *on* $\mathbb{T}$ *and* $\boldsymbol{u}$ *is a sequence in* $\mathbb{T}$,

$$K_{2n}(a_1, \ldots, a_{n-1}, u_n; z) \rightrightarrows F_\mu(z), \quad z \in \mathbb{C}\setminus \overline{\lim_n}\, \Sigma_n(\boldsymbol{a}; \boldsymbol{u}).$$

Moreover, each $z \in \overline{\lim}_n\, \Sigma_n(\boldsymbol{a}; \boldsymbol{u})\setminus \operatorname{supp}\mu$ (*up to, at most, one point*) *has a neighbourhood where the above uniform convergence holds at least for a subsequence.*

This corollary says that $(\Psi_n^{u_n}/\Phi_n^{u_n})_{n\geqslant 1}$ converges to F_μ, not only outside the unit circle, but also at the points in the unit circle that are not limit points of the zeros of the para-orthogonal polynomials $(\Phi_n^{u_n})_{n\geqslant 1}$. The results of the previous sections can now be used to get information about the convergence of the sequence $(\Psi_n^{u_n}/\Phi_n^{u_n})_{n\geqslant 1}$, as the following examples show.

Example 5.4. Let $\boldsymbol{a}$ be the sequence of Schur parameters of a measure μ on $\mathbb{T}$ and let $\boldsymbol{u}$ be a sequence in $\mathbb{T}$.

1. Schur parameters converging to the unit circle.
   If $\lim_n |a_n| = 1$ and $u_n = \frac{a_n}{|a_n|}$, Corollary 4.8 gives

$$K_{2n}(a_1, \ldots, a_{n-1}, u_n; z) \rightrightarrows F_\mu(z), \quad z \in \mathbb{C}\setminus \operatorname{supp}\mu.$$

2. Rotated asymptotically 2-periodic Schur parameters.
   Let $\lim_n a_{2n-1}(\lambda) = a_o$, $\lim_n a_{2n}(\lambda) = a_e$, $\lambda \in \mathbb{T}$. Example 4.9 shows that, if $u_n = \frac{a_n}{|a_n|}$, the conditions $\rho_o\rho_e \mp \mathrm{Re}(\overline{a}_o a_e) < \min\{|a_o|, |a_e|\}$, respectively, imply that

   $$K_{2n}(a_1, \ldots, a_{n-1}, u_n; z) \rightrightarrows F_\mu(z), \quad z \in \mathbb{C} \backslash (\mathrm{supp}\,\mu \cup \triangle_{\beta_\pm}(\pm\lambda)),$$

   where $\beta_\pm \in (0, \pi]$ are given by $\cos \frac{\beta_\pm}{2} = \sqrt{\frac{1+\rho_o\rho_e \mp \mathrm{Re}(\overline{a}_o a_e)}{1+\min\{|a_o|,|a_e|\}}}$.
3. The limit points of the odd and even subsequences of $\boldsymbol{a}(-\lambda)$, $\lambda \in \mathbb{T}$, separated by a band.
   If $\mathcal{B}(u, \alpha_1, \alpha_2)$, $u \in \mathbb{T}$, $0 \leqslant \alpha_1 < \alpha_2 \leqslant \pi$, is such a band, Theorems 3.4 and 4.11 prove that

   $$K_{2n}(a_1, \ldots, a_{n-1}, u_n^w; z) \rightrightarrows F_\mu(z), \quad z \in \mathbb{C} \backslash (\mathrm{supp}\,\mu \cup \triangle_\alpha(\lambda) \cup \{w\}),$$

   where $\alpha \in (0, \pi]$ is given by $\sin \frac{\alpha}{2} = \max\{\sin \frac{\alpha_2}{2} - \sin \frac{\alpha_1}{2}, \cos \frac{\alpha_1}{2} - \cos \frac{\alpha_2}{2}\}$ and w is arbitrarily chosen in $\Gamma_\alpha(\lambda)$.
4. $(\frac{a_{n+1}}{a_n})_{n \geqslant 1}$ converging to the unit circle.
   Let us suppose that $\mathfrak{L}\{\frac{a_{n+1}}{a_n}\} \subset \overline{\Gamma}_\zeta(\lambda)$, $\lambda \in \mathbb{T}$, $\zeta \in [0, \pi)$. Theorems 3.10, 4.11 and Corollary 3.9 imply that, if $\sin \dfrac{\zeta}{2} < \underline{\lim}_n |a_n|$, then

   $$K_{2n}(a_1, \ldots, a_{n-1}, u_n^w; z) \rightrightarrows F_\mu(z), \quad z \in \mathbb{C} \backslash (\mathrm{supp}\,\mu \cup \triangle_{\alpha-\zeta}(\lambda) \cup \{w\}),$$

   where $\alpha \in [0, \pi]$ is given by $\sin \frac{\alpha}{2} = \underline{\lim}_n |a_n|$ and w is any point in $\Gamma_{\alpha-\zeta}(\lambda)$.
   In particular, in the López class $\lim_n \frac{a_{n+1}}{a_n} = \lambda \in \mathbb{T}$, $\lim_n |a_n| \in (0, 1)$, we have $\zeta = 0$ and $\{\mathrm{supp}\,\mu\}' = \triangle_\alpha(\lambda)$, hence,

   $$K_{2n}(a_1, \ldots, a_{n-1}, u_n^w; z) \rightrightarrows F_\mu(z), \quad z \in \mathbb{C} \backslash (\mathrm{supp}\,\mu \cup \{w\}),$$

   if we choose $w \in \Gamma_\alpha(\lambda)$.

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